

What is a Knot?

A knot is a closed, non-self-intersecting loop in three-dimensional space, considered up to continuous deformations. In other words, a knot is like a piece of string that is tangled up and then has its ends joined together, forming a loop.

A mathematical knot is an embedding of the circle S^1 into three-dimensional space $\mathbb{R}^3$, where two knots are considered the same if one can be smoothly deformed into the other without cutting or passing through itself.

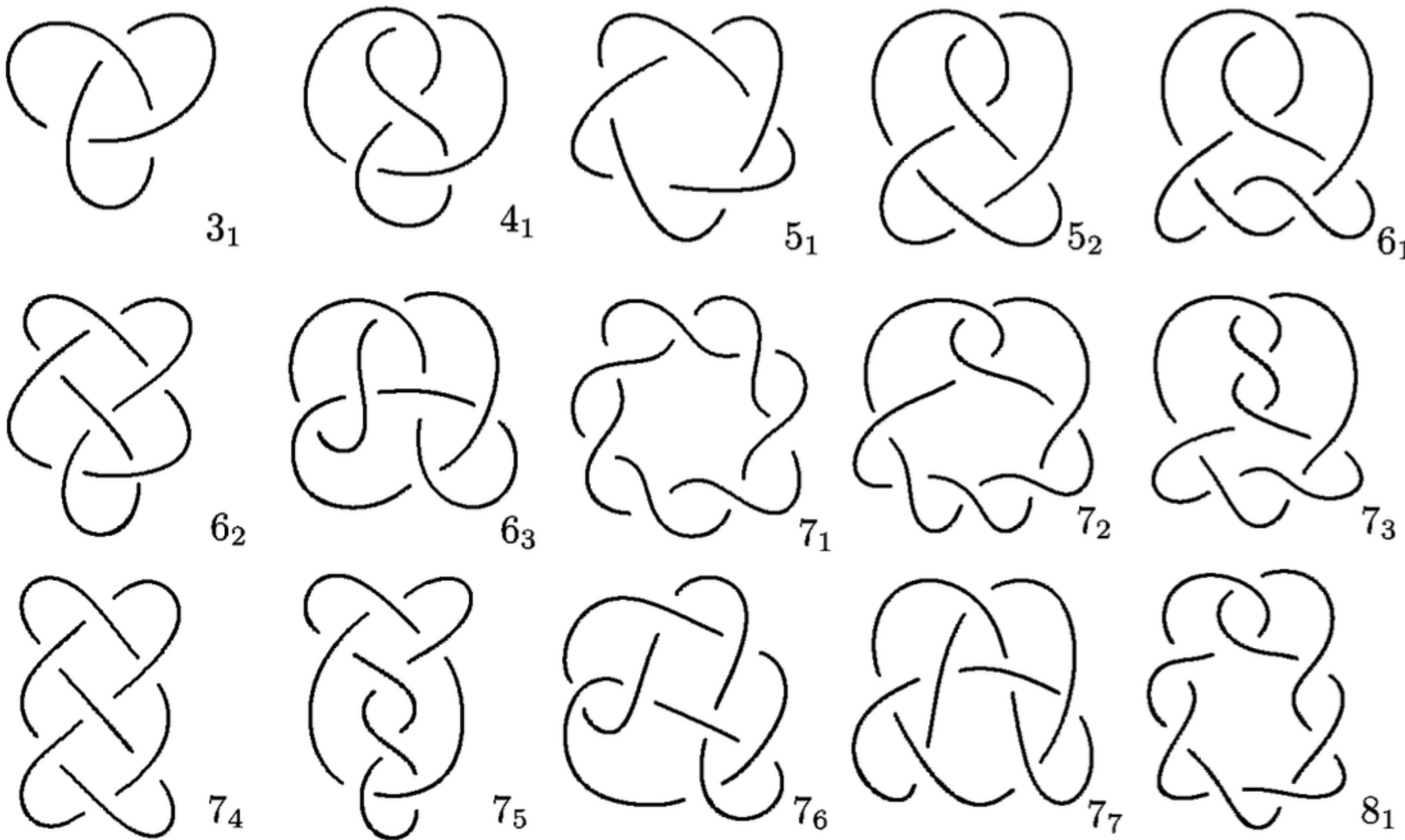


Figure 1. Knots with 3 - 8 crossings

Rock climbers use a variation of the figure-eight knot (4_1 knot in the diagram), called the figure-eight follow-through knot, as their primary way to tie into a climbing harness.

Orientation: Positive Crossing and Negative Crossing

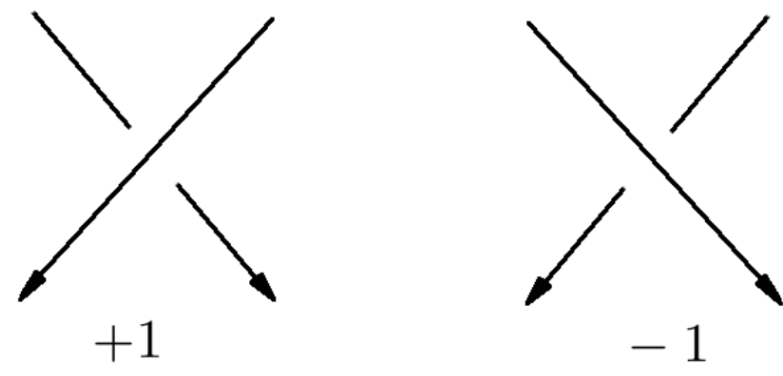


Figure 2. Sign of Crossings

What is a Quandle? Algebraic Definitions

A **quandle** is a set Q equipped with a binary operation $*$: $Q \times Q \rightarrow Q$ satisfying the following three axioms:

- Idempotency:** For all $x \in Q$, $x * x = x$. (when an element interacts with itself under the quandle operation, it remains unchanged).
- Right Invertibility:** For every $x, y \in Q$, there exists a unique $z \in Q$ such that $z * y = x$.
- Right Self-Distributivity:** For all $x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

Knot Invariant

A knot invariant is a property or quantity associated with a knot that remains unchanged under Reidemeister moves and continuous deformations (ambient isotopies). In other words, no matter how you stretch, twist, or manipulate a knot diagram (without cutting or passing strands through each other), the knot invariant stays the same.

- Crossing Number** – The smallest number of crossings in any diagram of the knot.
- Tricolorability** – A simple way to distinguish some knots using three colors.
- Fundamental Group (Knot Group)** – A topological invariant capturing how loops around the knot behave.
- Jones Polynomial** **Alexander Polynomial** – Algebraic invariants that assign polynomials to knots.
- Quandles** – A powerful invariant that encodes the structure of how arcs interact at crossings.

Reidemeister Moves

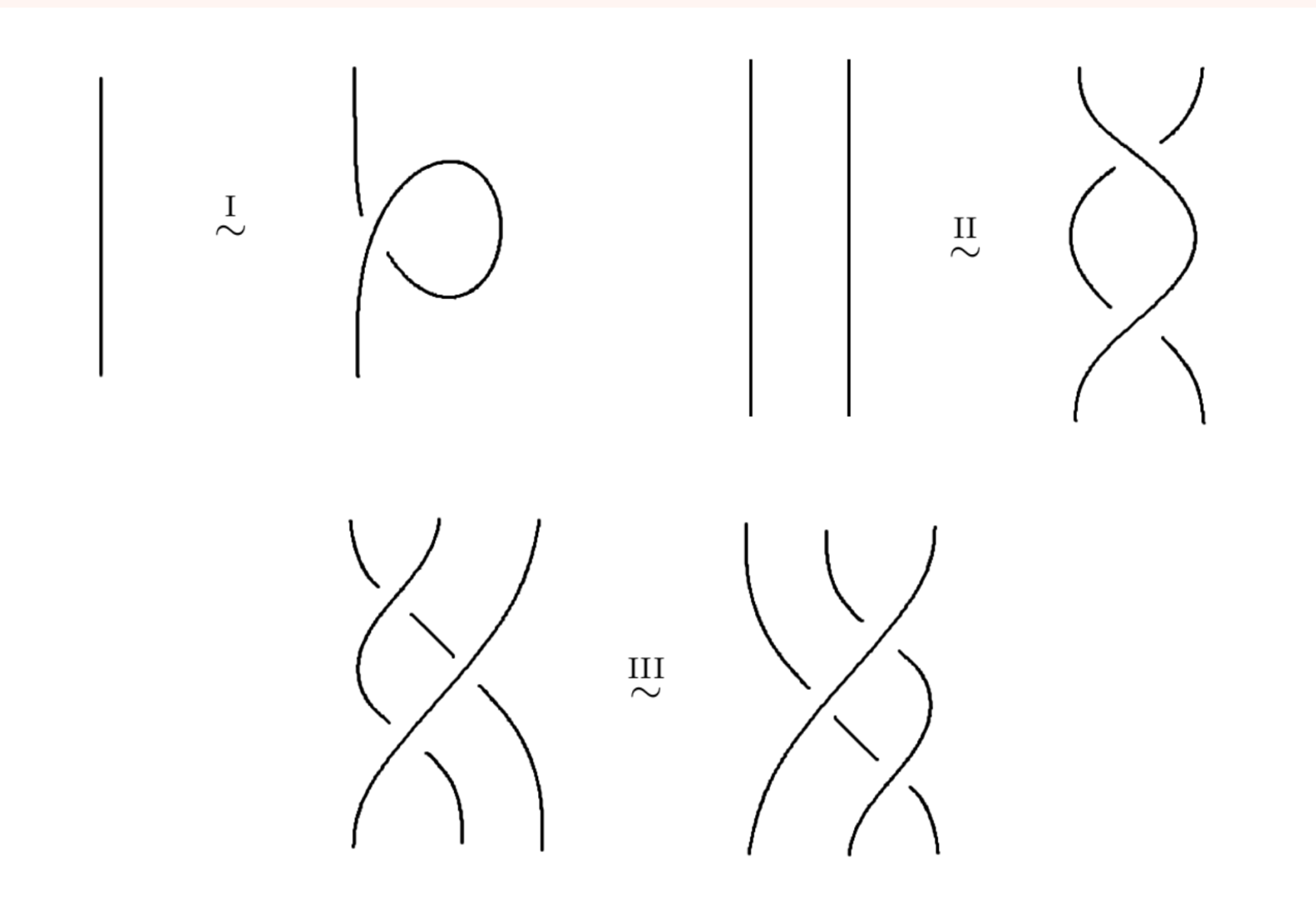


Figure 3. Reidemeister Moves I, II, III

Reidemeister moves are three basic diagram transformations that do not change the fundamental structure of a knot.

Any two diagrams of the same knot can be transformed into each other using only Reidemeister moves plus smooth deformations in space.

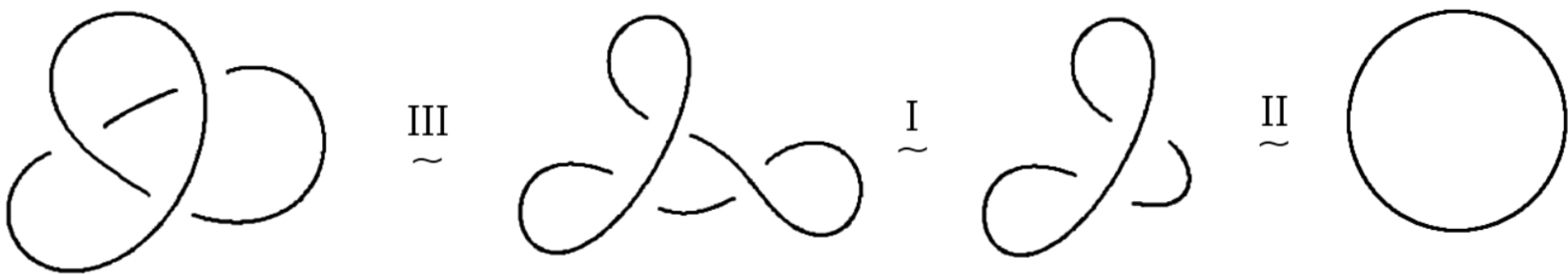


Figure 4. Reidemeister Moves Transformation

This process does not change the fundamental structure of this knot.

References

[1] Mohamed Elhamdadi and Sam Nelson. Quandles-an introduction to the algebra of knots. *American Mathematical Society*, 74(3):379–423, 2015.

Motivation of Quandles

Quandles are algebraic structures that help us distinguish knots (differentiate one knot from another).

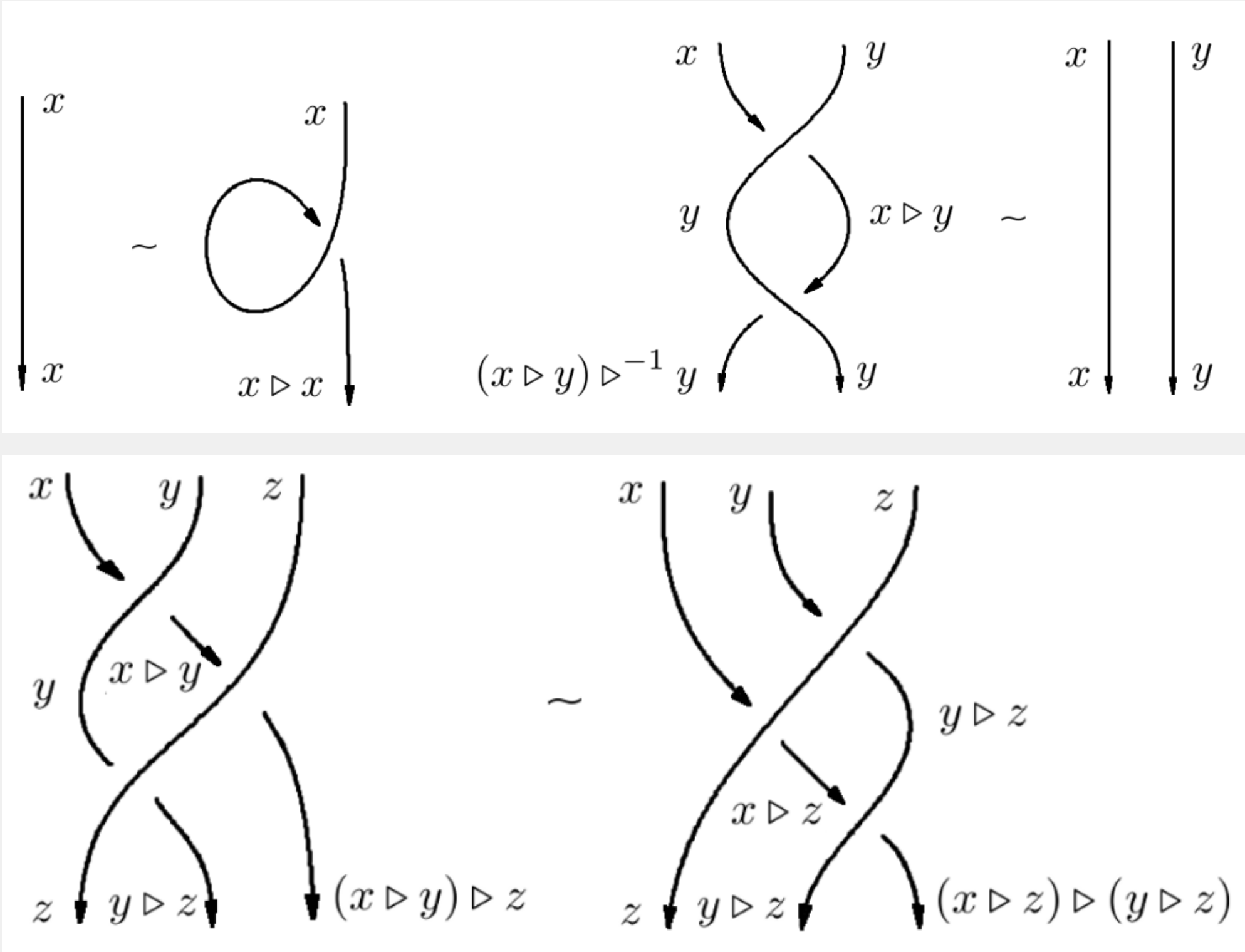
Reidemeister Moves Invariance

The three Reidemeister moves (which define when two diagrams represent the same knot) correspond exactly to the quandle axioms:

- Idempotency** ⇔ Type I move (loop addition/removal)
- Right Invertibility** ⇔ Type II move (strand swapping)
- Self-distributivity** ⇔ Type III move (strand sliding over a crossing)

Every knot has an associated fundamental quandle, which serves as a complete invariant.

Generalize Knot Coloring



Example - Alexander Quandle

An **Alexander quandle** is a quandle structure on an abelian group Q with a binary operation defined by: $x * y = tx + (1 - t)y$ where t is a fixed element of a ring, typically $\mathbb{Z}[t, t^{-1}]$. Its operation satisfies the three fundamental quandle axioms:

- Idempotency:** $x * x = tx + (1 - t)x = x$.
- Right Invertibility:** Given $x, y \in Q$, we need to solve for z such that: $z * y = x$.
Expanding: $tz + (1 - t)y = x$.
Solving for z : $z = t^{-1}x + (1 - t^{-1})y$.
If t is invertible, this uniquely defines z .
- Right Self-Distributivity:** We check:
 $(x * y) * z = t(x * y) + (1 - t)z$.
Simplifying:
 $t^2x + (1 - t)ty + (1 - t)z$.
Since both expressions are equal, self-distributivity holds.