

Phy 230 B Lecture 6

massive spin 1

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{8} (A^\mu A_\mu)^2$$

$$X \propto \lambda$$

$$m(A(p_1, h=0) A(p_2, h=0) \rightarrow A(p_3, h=0) A(p_4, h=0)) \sim \lambda \frac{E^4}{m_A^4}$$

$$[\lambda]=0, [m]=0 \Rightarrow m \propto \lambda \quad \text{not true, mass is somehow still important even in high energy limit}$$

high energy limit, mass doesn't matter bc energy is thousands of times higher

What about massless spin 1?

Gauge Invariance

Mysterious: $\partial_\mu A^\mu = 0$ "extra constraint"

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu + \frac{\alpha}{2} (\partial_\mu A^\mu)^2$$

Massless $\rightarrow$ simpler but weirder than massive particles

Particles with Spin

$$|p, s\rangle \quad p^2 = 0, s = \dots$$

$$p^\mu = k^\mu = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} \quad E > 0$$

particles at rest, but we're dealing with massless!

reference momentum

② $\Omega =$ Lorentz Transformation

$$\Omega \cdot k = k$$

$$\Omega = e^{-i\theta J^3} \quad \text{rotation around z-axis}$$

... but there's more Lorentz transformation that leaves k invariant.

$$J^3 = M^{12}$$

e.g. ①

$$\vec{k} \rightarrow \vec{k}'$$

boost along x_2

$$\vec{k} \rightarrow \vec{k}' \quad (\text{infinitesimal})$$

Rotation around x_1 , $\vec{k}' \rightarrow \vec{k}$ (infinitesimal)

generator 2

$$M^{02} + M^{23}$$

$$= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$T' \cdot k = 0$$

$$e^{i\vec{T} \cdot \vec{k}} = 0$$

e.g. ②

generator
3

$$T^2 = - (M^{01} + M^{13})$$

General transformation for little group:

$$\Omega = e^{-i(\epsilon^1 T^1 + \epsilon^2 T^2)} e^{-i\theta J^3}$$

(any generator
commutes
with itself)

$$[J^3, T^1] = +i T^2$$

$$[J^3, T^2] = -i T^1$$

$$[T^1, T^2] = 0$$

2D Euclidean Group

"ISO(2)"

Isometries of 2D plane: rotations + translations

J^3 : rotation generator

T^1, T^2 : translation generator

Find Unitary Representations of ISO(2)

faithful

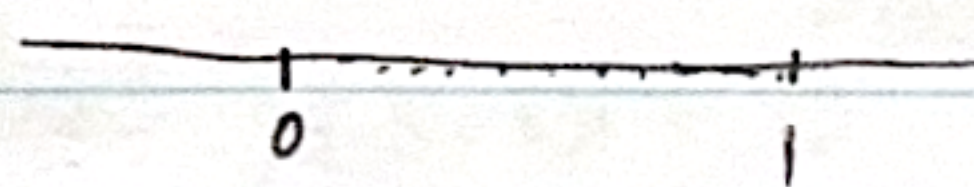
Lie

Fact: Unitary representations of non-compact groups are ∞ dimensional

non-compact: idea: "space that goes on forever"

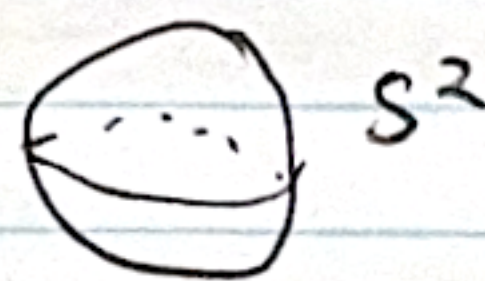
precise: "space doesn't contain all its limit points"

ex: $[0, 1]$ - compact



$\mathbb{R}$ - non-compact

$SO(3)$ - compact group

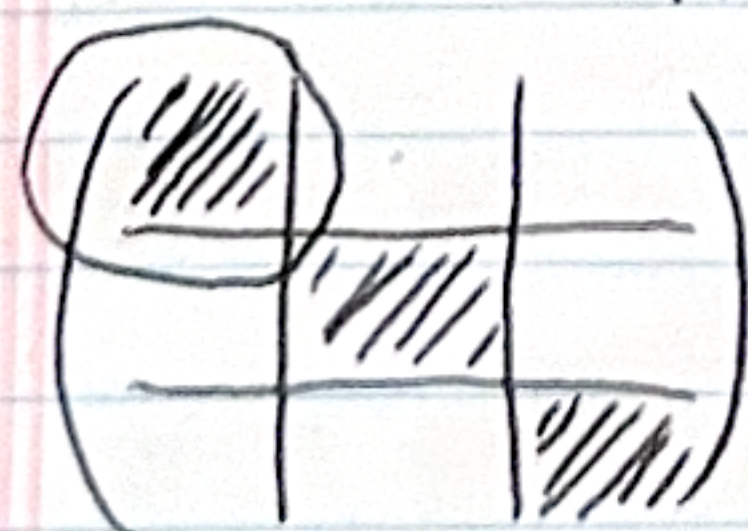


boost $e^{-i\eta M^{02}}$ - non-compact

Lorentz Group is non-compact (we're okay with ∞ dimensional unitary rep)

ISO(2) - also noncompact

acts on $|k, s\rangle \rightarrow |k, s'\rangle \times s's$



$\Rightarrow$ seems to imply ∞ spin states ???

$\hat{T}^{1,2} |k, s\rangle = 0$ (invariant under translations)

so only $SO(2) \subset ISO(2)$ acts nontrivially

$$e^{-i(\epsilon^1 T^1 + \epsilon^2 T^2)} = 1 + i(\epsilon^1 T^1 + \dots) + \dots$$

$$|k, s\rangle = |k, s\rangle$$

$$|k, s\rangle \mapsto |k, s\rangle + \underbrace{i(\xi^1 \hat{T}^1 + \xi^2 \hat{T}^2)}_{=0} |k, s\rangle + O(\xi^2)$$

$$\boxed{\hat{T}^{1,2} |k, s\rangle = 0} \rightarrow \text{infinitesimal variation}$$

If we want finitely many spin states, then we need

Unitary reps of the little group:

$$|k, s\rangle \mapsto \underbrace{e^{-i\Theta \hat{J}^3}}_{= e^{-i s \Theta}} |k, s\rangle$$

define s to be eigenvalue of J^3

require it = 1 for $\Theta = \pm 2\pi, \pm 4\pi, \dots$

$$\Rightarrow s = 0, \pm 1, \pm 2, \dots$$

for fermions = -1 , $s = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$

unitary rep is just 1 state:

$$s = 0, \pm \frac{1}{2}, \pm 1, \dots$$

$|k, +1\rangle; |k, -1\rangle$ these guys come in pairs

different representation of little group

$$\hat{CPT} |k, +1\rangle = |k, -1\rangle$$

$$\hat{U}(e^{-i(\xi^1 \hat{T}^1 + \xi^2 \hat{T}^2)} e^{-i\Theta \hat{J}^3}) |k, s\rangle = e^{-i\Theta s} |k, s\rangle$$

massless: $\hat{U}(\Lambda) |p, s\rangle = |\Lambda p, s\rangle e^{-i\Theta(\Lambda, p)}$ ← weird phase

massive: $= |\Lambda(p), s'\rangle D_{s's}^{(s)}(W(\Lambda, p))$

$\Theta(\Lambda, p)$ Wigner rotation

$$W(\Lambda, p) \stackrel{\text{def}}{=} L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p)$$

$$W(\Lambda, p) \cdot k = k \quad (\text{true in massless case})$$

$$W(\Lambda, p) = e^{-i(\xi^1 \hat{T}^1 + \xi^2 \hat{T}^2)}_{(\Lambda, p)} e^{-i\Theta(\Lambda, p)}$$

We never need to know explicit $\Theta(\Lambda, p)$

$s = \text{helicity}$

$$\hat{h}(p) \stackrel{\text{def}}{=} \frac{\vec{p} \cdot \vec{J}}{|\vec{p}|}$$

$\hat{h}(p) |p, s\rangle = s |p, s\rangle$ for any p
(any direction)
 $\Rightarrow$ helicity is a Lorentz invariant quantity

(angular momentum conserved globally)
Massive vs. Massless Spin 1

$$m \neq 0: |p, h\rangle \quad h = 0, \pm 1$$

$$m = 0: |p, h\rangle \quad h = \pm 1 \quad (\text{they have to come in pairs})$$

$$\langle p, h | \hat{A}^\mu(0) | \Omega \rangle \neq 0? \quad \mu = 0, 1, 2, 3$$

massless
doesn't work !!!

$\rightarrow$ gauge inv. fixes this
loop hole
(for next time)