

Phy 230 B Lecture 5

Adding Interactions

$$\mathcal{L} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu}_{\text{free}} - \frac{\lambda}{8} (A^\mu A_\mu)^2$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

$$\Rightarrow \partial_\mu A^\mu = 0$$

$$m^2 A^\nu + \partial_\mu F^{\mu\nu} = \frac{\lambda}{4} (A^\mu A_\mu) A^\nu$$

$$\partial_\nu [] = 0 + m^2 \partial_\nu A^\nu = \frac{\lambda}{4} \partial_\nu [A^\mu A_\mu A^\nu] \neq 0$$

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$$\hat{H}_0 = \int (dp) \sum_{s=0, \pm 1} E_p \hat{a}_s^\dagger(p) \hat{a}_s(p)$$

independent of time → Schrödinger Picture

$$\underbrace{\hat{H}_{int}}_{\text{S.P.}} \stackrel{\text{def}}{=} \int d^3x \frac{\lambda}{8} \underbrace{[\hat{A}^\mu(\vec{x}) \hat{A}_\mu(\vec{x})]^2}_{\text{S.P.}}$$

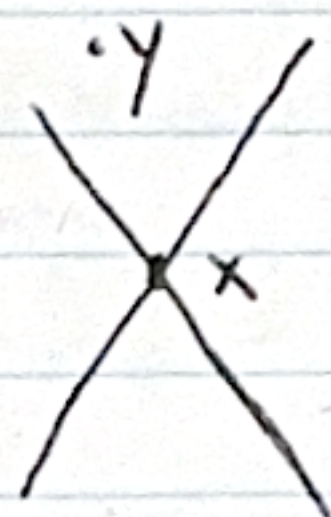
interaction picture evolve

$$\hat{A}_{IP}^\mu(x) = \int (dp) \left[\sum_s \hat{a}_s(p) \epsilon_s^\mu(p) e^{-ip \cdot x} + \text{h.c.} \right]$$

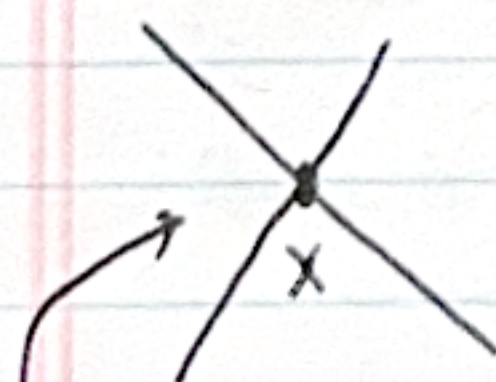
interaction picture time evolution operator from $t \in (-\infty, \infty)$

$$\hat{S} = T \exp \left\{ - \int_{-\infty}^{+\infty} dt \hat{H}_{int, IP}(t) \right\} = T \exp \left\{ -i \int d^4x \frac{\lambda}{8} A_{IP}^4 \right\}$$

Lorentz Invariant



: all observers agree x can influence y



y : can both happen first

$$T[\hat{A}^\mu(x), \hat{A}^\nu(y)] = 0?$$

yes $[\hat{A}^\mu(x), \hat{A}^\nu(y)] = 0$ x-y space-like separated

$$\begin{aligned} \langle f | \hat{S} | i \rangle &= \langle f | i \rangle \\ &\quad - i \langle f | \textcircled{T} \int \frac{\lambda}{8} A^4 | i \rangle, \\ &\quad + \frac{(-i)^2}{2!} \langle f | \textcircled{T} \left\{ \int \frac{\lambda}{8} A^4 \int \frac{\lambda}{8} A^4 \right\} | i \rangle \\ &\quad + \dots \end{aligned}$$

Wick's Theorem:

$$T[\dots] = \text{:} \dots \text{:} \text{ normal ordering} \\ + \text{contractions}$$

$$T[\phi_1, \phi_2, \phi_3, \phi_4] = \text{:} \phi_1 \phi_2 \phi_3 \phi_4 \text{:} + \text{:} \phi_1 \phi_2 \text{:} \phi_3 \phi_4 + \text{:} \phi_1 \phi_3 \text{:} \phi_2 \phi_4 + \text{:} \phi_1 \phi_4 \text{:} \phi_2 \phi_3 + \text{perms},$$

$$\underbrace{\hat{\phi}(x) \hat{\phi}(y)} = \Delta_F(x-y)$$

+ perms.

Feynman Propagator

normal ordered $\rightarrow$

$$\langle f | \text{:} \dots \text{:} | i \rangle = \text{contractions between fields and particles}$$

$$\langle 0 | \hat{a} \dots \hat{a} \text{:} \dots \text{:} \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger | i \rangle$$

incoming

$$\text{:} \dots \hat{\phi}(x) \dots \text{:} | \dots \phi(p) \dots \rangle = e^{-ip \cdot x}$$

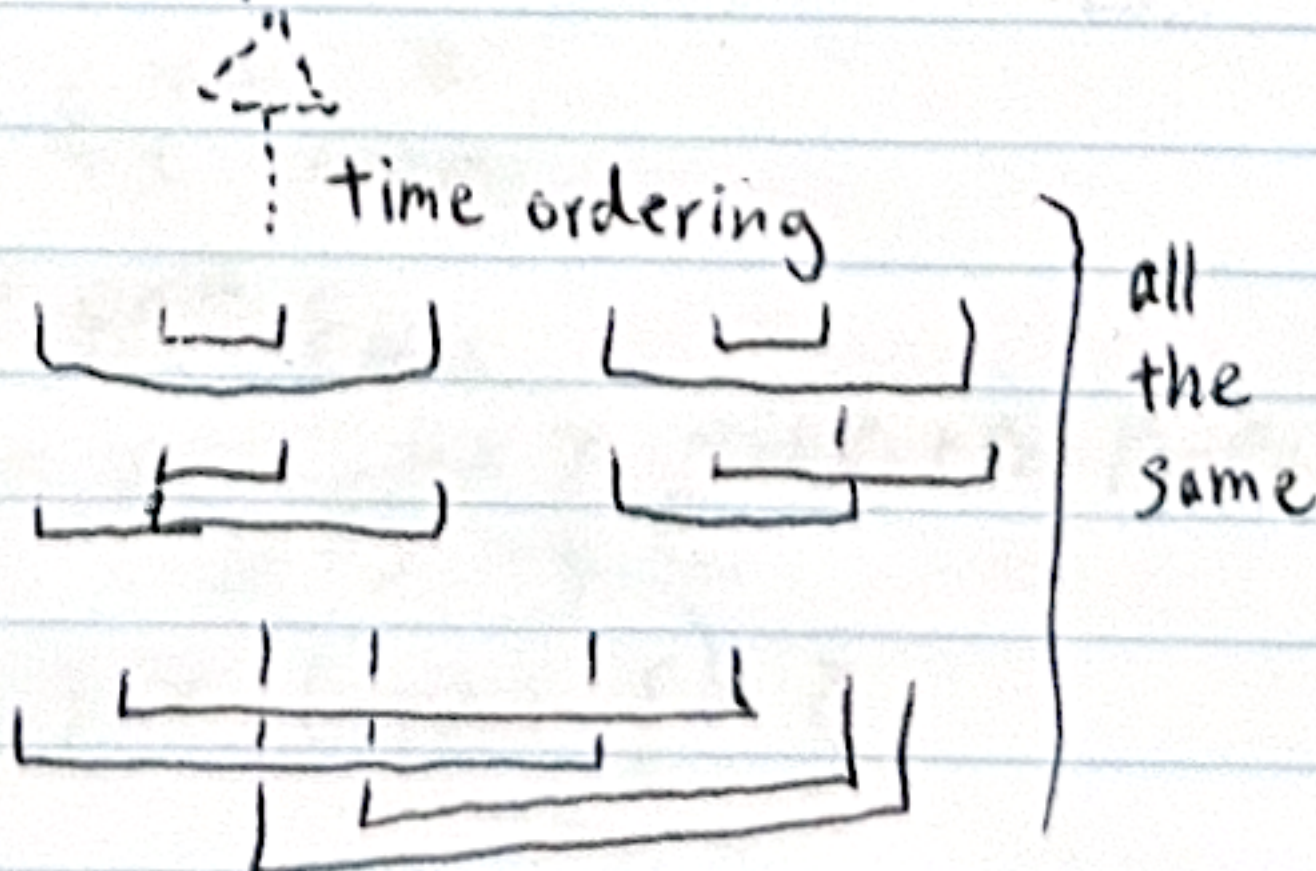
$$| \rangle = | A(p_1, h_1) A(p_2, h_2) \rangle$$

$$f \rangle = | A(p_3, h_3) A(p_4, h_4) \rangle \quad h_i = 0, \pm 1$$

$$S_{fi} = -i \frac{\lambda}{8} \int d^4x \langle A_3 A_4 | \hat{A}^\mu \hat{A}_\mu \hat{A}^\nu \hat{A}_\nu | A_1 A_2 \rangle + O(\alpha^2)$$

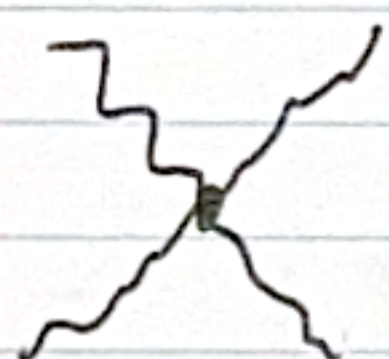
$$\underbrace{\langle A_3 A_4 | \hat{A}^\mu \hat{A}_\mu \hat{A}^\nu \hat{A}_\nu | A_1 A_2 \rangle}$$

This is different

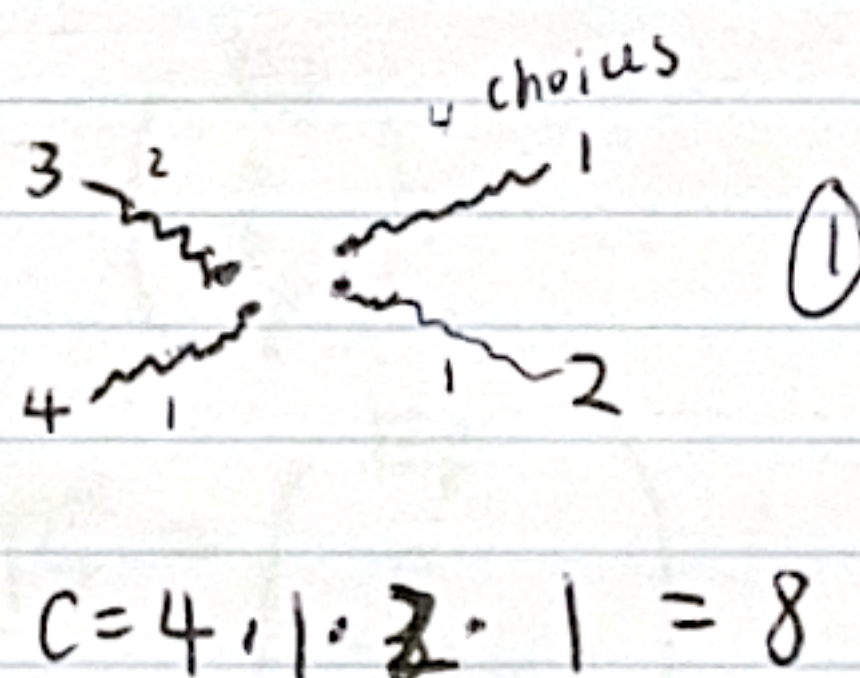
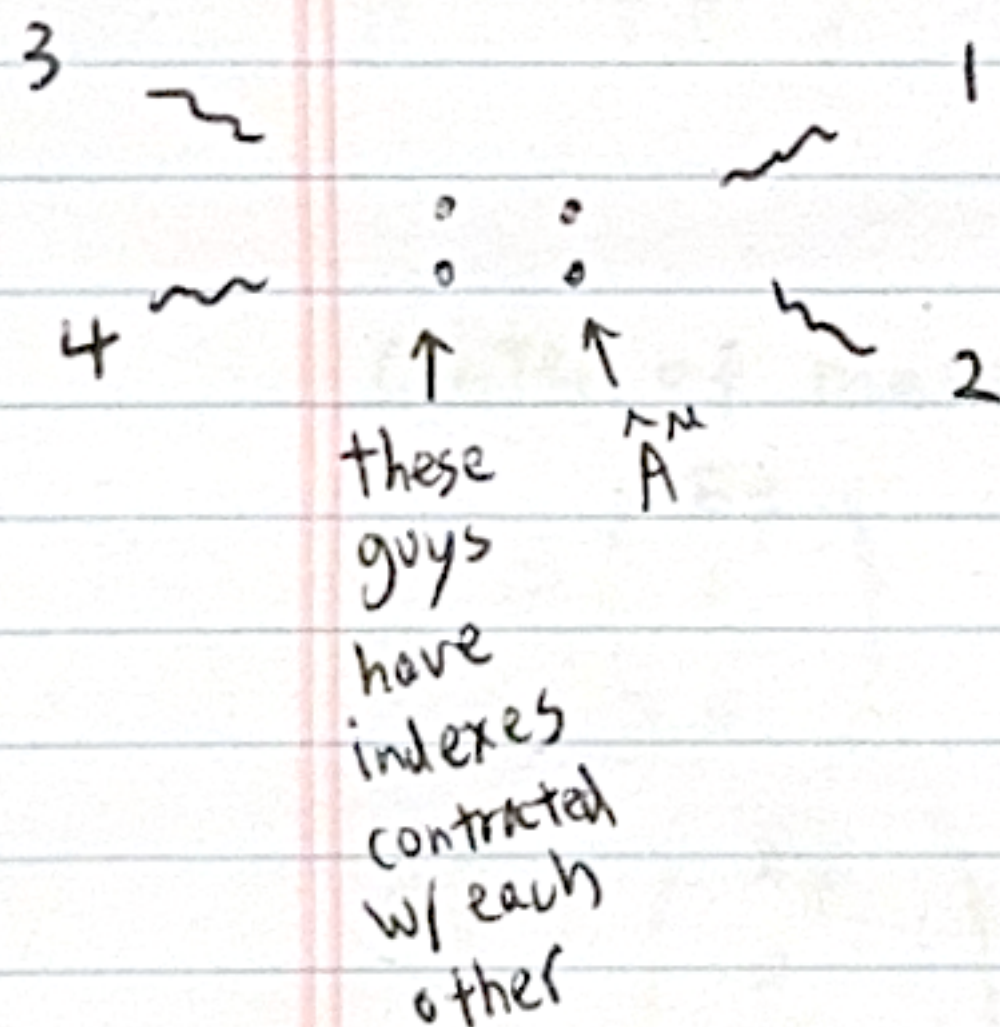


Classify contractions into equiv. classes

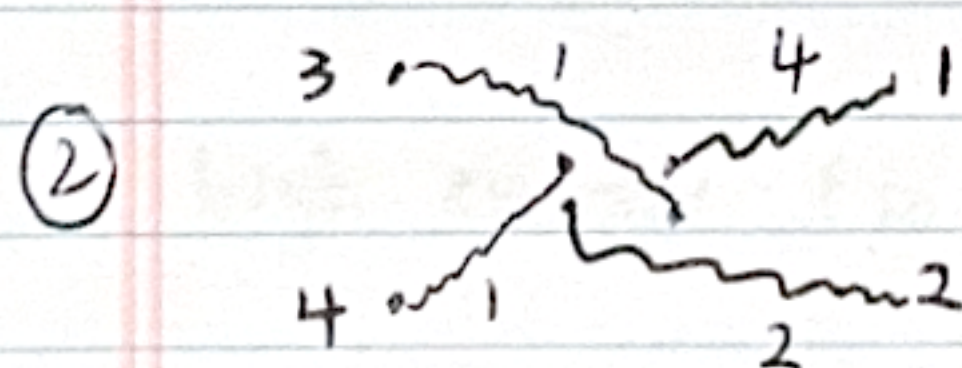
$$S_{fi} = -i \frac{\lambda}{8} \int d^4x \left\{ \underbrace{\langle A_3 A_4 |}_{\text{C}} \underbrace{\hat{A}^\mu \hat{A}_\mu \hat{A}^\nu \hat{A}_\nu}_{\text{C'}} | A_1 A_2 \rangle \cdot \text{C} \right. \\ \left. + \underbrace{\langle A_3 A_4 |}_{\text{C'}} \underbrace{\hat{A}^\mu \hat{A}_\mu \hat{A}^\nu \hat{A}_\nu}_{\text{C''}} | A_1 A_2 \rangle \cdot \text{C''} \right\}$$



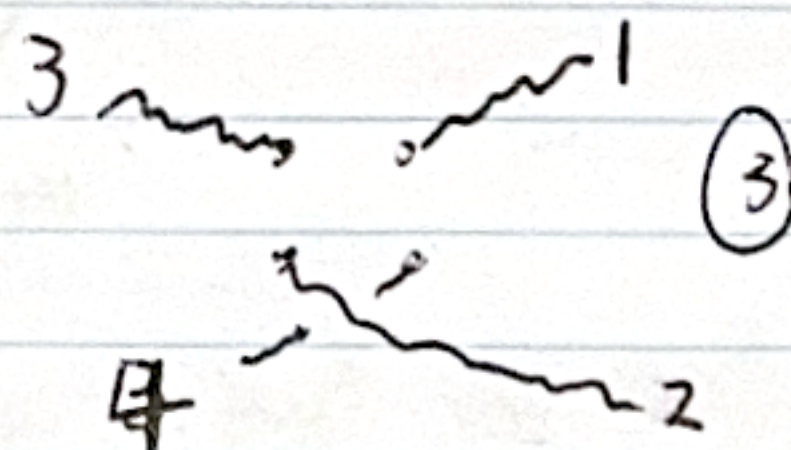
$$\dots \hat{A}^\mu(x) \dots = \underbrace{\epsilon_S^\mu(p) e^{-ip \cdot x}}_{\text{switching : with ; doesn't do anything}}$$



$$C = 4 \cdot 1 \cdot 2 \cdot 1 = 8$$



$$C' = 4 \cdot 2 \cdot 1 \cdot 1 = 8 \quad \epsilon_{S_2}^\nu(p_2)$$



$$C'' = 8$$

$$S_{fi} = -i \frac{\lambda}{8} \cdot 8 \int d^4x \left\{ \epsilon_{1\nu} \epsilon_2^\nu \epsilon_3^{\mu*} \epsilon_{4\mu}^* \right. \\ \left. + \epsilon_{1\nu} \epsilon_3^\nu \epsilon_2^{\mu*} \epsilon_{4\mu}^* \right. \\ \left. + \epsilon_{1\nu} \epsilon_4^{\nu*} \epsilon_{2\mu} \epsilon_3^{\mu*} \right\} e^{-i(p_1 + p_2 - p_3 - p_4) \cdot x}$$

$$S_{fi} = (-2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times (-i\lambda) \{ \}$$

M_{fi} scattering amplitude

$$M_{fi} = -i\lambda \left(\epsilon_1 \cdot \epsilon_2 \epsilon_3^* \cdot \epsilon_4^* + \epsilon_1 \cdot \epsilon_3^* \epsilon_2 \epsilon_4^* + \epsilon_1 \cdot \epsilon_4^* \epsilon_2 \cdot \epsilon_3^* \right) + O(\lambda^2)$$

$$m \left(A(p_1, 0) A(p_2, 0) \rightarrow A(p_3, 0) A(p_4, 0) \right)$$

$\uparrow$
 $h=0$

$$\epsilon_{h=0}^\mu(p) = \frac{1}{m} \begin{pmatrix} |\vec{p}| \\ \frac{E_{\vec{p}}}{|\vec{p}|} \cdot \vec{p} \end{pmatrix} \quad \text{spatial pointed to momentum}$$

$$p^\mu = \begin{pmatrix} E_{\vec{p}} \\ \vec{p} \end{pmatrix}$$

$$\epsilon_{h=0}(p) \parallel \vec{p}$$

$$p \cdot \epsilon_h(p) = 0$$

$$\epsilon_h(p) \cdot \epsilon_h(p)^* = -1$$

center of mass frame

$$p_1^\mu = \begin{pmatrix} E_{\vec{p}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad p_2^\mu = \begin{pmatrix} E_{\vec{p}} \\ 0 \\ 0 \\ -p \end{pmatrix}$$

$$p_3^\mu = \begin{pmatrix} E_{\vec{p}} \\ 0 \\ p \sin \theta \\ p \cos \theta \end{pmatrix} \quad p_4^\mu = \begin{pmatrix} E_{\vec{p}} \\ 0 \\ -p \sin \theta \\ -p \cos \theta \end{pmatrix}$$

high energy (in cm) $|\vec{p}| \gg m$

$$s = (p_1 + p_2)^2 \stackrel{\text{cm}}{=} \begin{pmatrix} 2E_{\vec{p}} \\ 0 \\ 0 \\ 0 \end{pmatrix}^2 = 4E_{\vec{p}}^2 \gg m^2$$

$$|\vec{p}| \simeq |E_{\vec{p}}| \gg m$$

high energy
limit

$$\epsilon_0^\mu(p) \rightarrow \frac{p^\mu}{m}$$

$$M_{00 \rightarrow 00} = \frac{-i\lambda}{m^4} [p_1 \cdot p_2 \cdot p_3 \cdot p_4 + p_1 \cdot p_3 \cdot p_2 \cdot p_4 + p_1 \cdot p_4 \cdot p_2 \cdot p_3]$$

$$\propto \frac{\lambda}{m^4} (s^2 + t^2 + u^2)$$

$$\sim \lambda \frac{E^4}{m^4}$$

dimension analysis

$$[m] = 0 \quad [\lambda] = 0$$

$$[m] = 1 \quad [E] = 1$$

↑
of a particle

$$m = \lambda (\dots) \quad \lambda \left(\frac{E}{m} \right)^{\#} \quad \# \downarrow \text{some power}$$

↑
dimensionless

power = 4 is a surprise

high energy $\Leftrightarrow m \rightarrow 0$

(fix m)

(fixed energy E)

$\lambda \left(\frac{E}{m} \right)^4$ diverges in high energy limit

a sign mass of particle

doesn't decouple in high energy limit (helicity 0)

$$M(A(p_1, 1) A(p_2, 1) \rightarrow A(p_3, 1) A(p_4, 1))$$

$\sim \lambda \left(\frac{E}{m} \right)^{\# = 0}$: mass is not important for helicity 1