

Phy 230 B Lecture 4

sign drops out for eom

at most 2 derivative
quadratic in the field

$$L = \pm \frac{1}{2} [\partial^\mu A^\nu \partial_\mu A_\nu + 2(\partial^\mu A_\mu)^2 - M^2 A^\mu A_\mu] \leftarrow O(\partial^2, A^2)$$

$$\delta A_\mu \Rightarrow \square A^\mu + 2 \partial^\mu \partial_\nu A^\nu + M^2 A^\mu = 0 \quad [\alpha] = 0 \quad [m^2] = 2$$

$$\partial_\mu A_\nu \partial^\mu A^\mu = (\partial_\mu A^\mu) \partial_\nu A^\nu + \text{total derivative}$$

↑
vanish
doesn't change
local EoM

Plane Wave Solution:

polarization vector, 4 of them

$$A^\mu(x) = \text{Re}[\epsilon^\mu(p) e^{-ip \cdot x}] \quad \partial_\mu \rightarrow -ip_\mu$$

$$-p^2 \epsilon^\mu - 2 p^\mu p_\nu \epsilon^\nu + M^2 \epsilon^\mu = 0 \quad \dots \textcircled{1}$$

$$[\text{scalar}: (p^2 - m^2) = 0 \Rightarrow p^2 = m^2]$$

$$S = \int d^4x \partial_\mu k^\mu$$

$$\delta S = \int d^4x \partial_\mu (S k^\mu)$$

"Transverse" polarization: $p_\mu \epsilon^\mu_\perp(p) \stackrel{\text{def}}{=} 0$

Then $\textcircled{1}$ becomes $[p^2 - m^2] \epsilon^\mu_\perp(p) = 0$

because $-2 p^\mu p_\nu \epsilon^\nu$ drops out

$$\textcircled{1} \Rightarrow p^2 = m^2 > 0 \quad p^\mu \rightarrow k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

↑
time-like vector

$$\epsilon^\mu_\perp(k) = \begin{pmatrix} 0 \\ \vec{e}_\perp \end{pmatrix} \Rightarrow 3 = (4-1) \text{ comp's}$$

general vector because $\epsilon^\mu_\perp(p) = 0$ only constrains 1
polarization transforms like a vector in rest frame
corresponds to spin 1 particles

"Longitudinal Polarization" $\epsilon^\mu_\parallel(p) = C p^\mu \quad C: \text{constant}$

$$\Rightarrow \text{eom}: [(1+\alpha)p^2 - m^2] \epsilon^\mu_\parallel(p) = 0 \quad ((\alpha \partial^\mu \partial_\nu A^\nu) \text{ doesn't drop out})$$

$$p^2 = \frac{m^2}{1+\alpha} > 0 = \tilde{m}^2 > 0$$

$$p^\mu \rightarrow \tilde{k}^\mu = \begin{pmatrix} \tilde{m} \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{different mass}$$

$$\epsilon^\mu_\parallel(\tilde{k}) = C \tilde{m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

invariant under rotations in rest frame

→ spin 0 particle

A field is seemingly describing 2 particles! but it doesn't work for both

Why it doesn't: Positivity of Energy

($\pm \frac{1}{2}$ doesn't drop out here)

work

$$T_{\nu}^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A^{\rho})} \partial_{\nu} A^{\rho} - \delta_{\nu}^{\mu} \mathcal{L}$$

energy
momentum
tensor

have to sum over
possible fields

$$T_{\mu\nu} = \pm \left[\partial_{\mu} A^{\rho} \partial_{\nu} A_{\rho} + \alpha (\partial_{\nu} A_{\mu}) (\partial^{\rho} A_{\rho}) \right] - \eta_{\mu\nu} \mathcal{L}$$

$$A^{\mu}(x) = \epsilon^{\mu}(p) \cos(p \cdot x) \quad \text{a soln that's real}$$

average
over 1
period

$$\langle T_{\mu\nu} \rangle = \pm \frac{1}{2} \left[p_{\mu} p_{\nu} \epsilon^2 + \alpha p_{\mu} \epsilon_{\nu} - \frac{1}{2} \eta_{\mu\nu} (p^2 - m^2) \epsilon^2 + \frac{1}{2} \alpha \eta_{\mu\nu} (p \cdot \epsilon)^2 \right]$$

rest
frame

$$\cdot \epsilon_{\perp}(p) \cdot p = 0 \quad \left. \vphantom{\begin{matrix} \cdot \epsilon_{\perp}(p) \cdot p = 0 \\ \cdot \epsilon_{\parallel}(p) \propto p^{\mu} \\ = c p^{\mu} \end{matrix}} \right\} p^{\mu} \rightarrow k^{\mu}$$

$$\cdot \epsilon_{\parallel}(p) \propto p^{\mu} = c p^{\mu} \quad \left. \vphantom{\begin{matrix} \cdot \epsilon_{\perp}(p) \cdot p = 0 \\ \cdot \epsilon_{\parallel}(p) \propto p^{\mu} \\ = c p^{\mu} \end{matrix}} \right\} p^{\mu} \rightarrow \tilde{k}^{\mu}$$

different
rest
frame

$$\Rightarrow \langle T_{00} \rangle = \pm \frac{1}{2} M^2 \epsilon^2$$

density of
energy in
rest frame

both $\epsilon_{\perp}$ and $\epsilon_{\parallel}$

$$\epsilon_{\perp}^2 < 0 \quad \text{choose - sign for } \epsilon_{\perp}^2 > 0$$

then $\epsilon_{\parallel}^2 > 0 \Rightarrow \langle T_{00} \rangle < 0$, Contradiction because we can't have positive energy for both

The real reason we can't have perpetual motion machine is bc energy is bounded below



Theoretically we can but
Realistically our world doesn't
allow negative energy

Coupling: positive energy $\uparrow \Rightarrow$ negative energy $\uparrow$

spin 0: $\tilde{m}^2 = \frac{M^2}{1+\alpha}$

$\alpha \rightarrow -1$ then mass goes to ∞
takes huge energy to excite it

(take a hint and)

call it m

Set $\alpha = -1$: $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu$

poca
Lagrangian

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ "field strength", anti-symmetric

$(\partial^\mu A^\nu \partial_\mu A_\nu + 2(\partial^\mu A_\mu)^2) \rightarrow F^{\mu\nu}$

$\delta A_\mu \Rightarrow \delta \mathcal{L} = -\frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu} + m^2 A^\mu \delta A_\mu$
 $= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$

$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ still 4 eom...

$\partial_\nu [\partial_\mu F^{\mu\nu} + m^2 A^\nu] = 0 + m^2 \partial_\nu A^\nu$
because $F^{\mu\nu}$ antisymmetric

$\Rightarrow \partial_\mu A^\mu = 0$ (consistency of eom implies)

$A^\mu(x) = \text{Re} [\epsilon^\mu(p) e^{-ip \cdot x}] \Rightarrow p \cdot \epsilon^\mu(p) = 0$
eliminates $\epsilon_{11}^\mu(p)$

$\partial_\mu A^\mu = \text{Re} [\epsilon^\mu(p) (-ip_\mu) e^{-ip \cdot x}] = 0$
 $\neq 0$

→ to interacting theory has to be 0 (we're heading to gauge thg)

Free Quantum Theory:

$\hat{A}^\mu(x) = \int (dp) \sum_{s=0,\pm 1} [\hat{a}_s(p) \epsilon_s^\mu(p) e^{-ip \cdot x} + \text{h.c.}]$

$\hat{a}_s^\dagger(p) |0\rangle = |A(p,s)\rangle$

$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$

$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

free Hamiltonian

$\hat{H}_0 = \int (dp) \sum_s E_{\vec{p}} \hat{a}_s^\dagger(p) \hat{a}_s(p) = \hat{p}^0$

adds up energy for all particles

$\hat{p}^\mu = \int (dp) \sum_s p^\mu \hat{a}_s^\dagger(p) \hat{a}_s(p)$

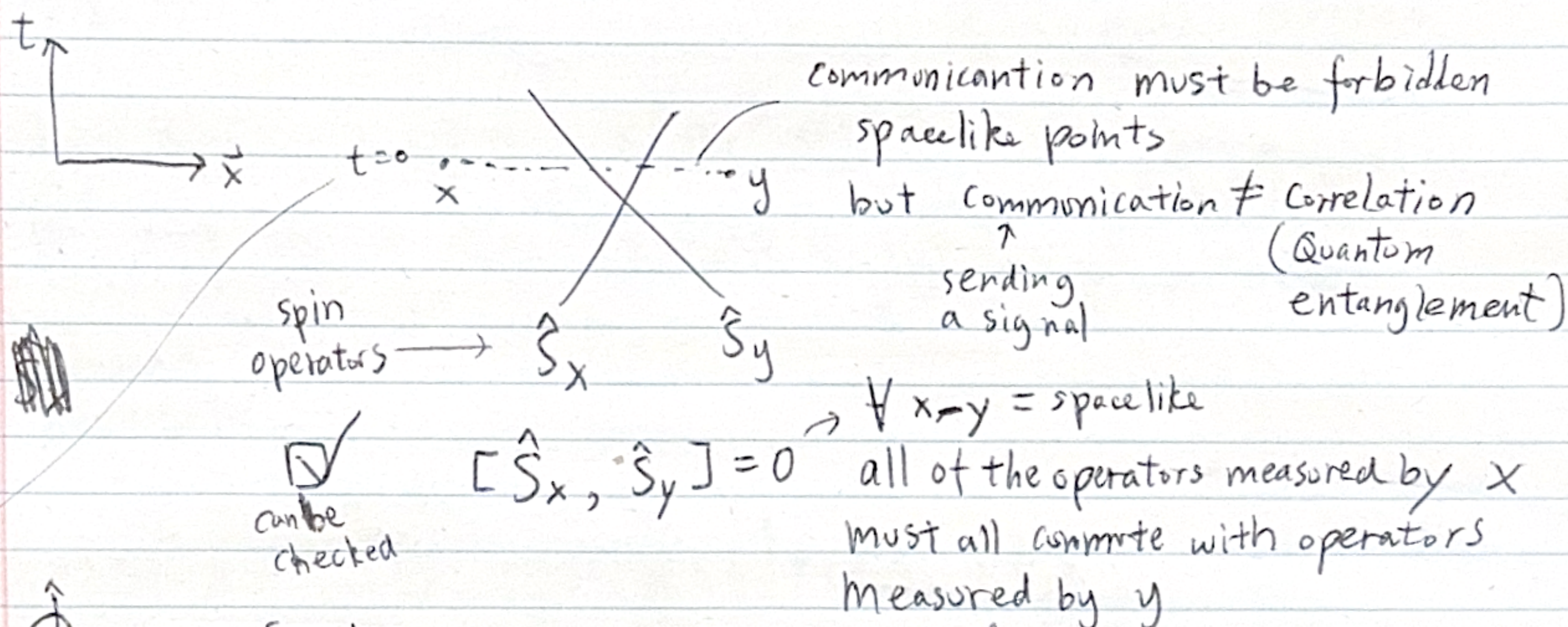
$\hat{p}^\mu |A(p,s)\rangle = p^\mu |A(p,s)\rangle$

different $\hat{H}_{int}$ describes diff. int. theory

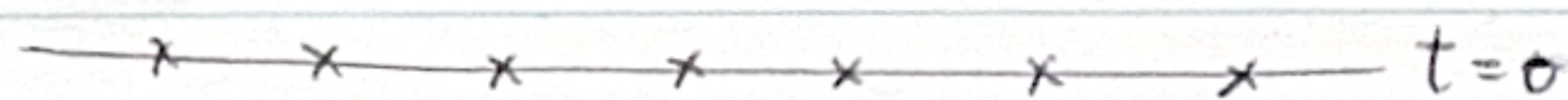
Claim: $\hat{H}_{int} = \int d^3x \frac{\lambda}{8} (\hat{A}^\mu \hat{A}_\mu)^2$ ($\hat{H}_{int} = \int d^3x \frac{\lambda}{4!} \hat{\phi}^4$)

$\Rightarrow \hat{S} = T \exp \left\{ -i \int d^4x \frac{\lambda}{8} (\hat{A}^\mu \hat{A}_\mu)^2 \right\} \leftarrow$ Lorentz Invariant S matrix

galilean boost $\rightarrow$ Lorentz boost
quantum mechanics $\rightarrow$ QFT



QFT $\hat{\phi}(x)$: a field is an operator that commute for all x
If there's interaction in Lagrangian then experimentalists can measure it



every point on slice has to have its own set of operators

Expand $\hat{S}$ in powers of λ

$\hat{S} = 1 + O(\lambda)$ $\leftarrow$ 0 for $i \neq f$

$\langle f | \hat{S} | i \rangle = \langle f | i \rangle - i \int d^4x \frac{\lambda}{8} \langle f | T (\hat{A}^\mu \hat{A}_\mu)^2 | i \rangle + O(\lambda^2)$

$|i\rangle = |A(p_1, h_1) A(p_2, h_2)\rangle$

$|f\rangle = |A(p_3, h_3) A(p_4, h_4)\rangle$

time ordered
specific spin initial and final states
 $\Rightarrow$ use h
more general spin
 $\Rightarrow$ use S



Wick's
Thm

$\langle f | T[\dots] | i \rangle =$ sum of all complete contractions

$\hat{A}^\mu(x) | \dots A(p, h) \dots \rangle$

contraction

$= \epsilon_h^\mu(p) e^{-ip \cdot x}$

$\hat{\phi}(x) | \dots \phi(p) \dots \rangle = \hat{a}(p) | \dots \phi(p) \dots \rangle = e^{-ip \cdot x}$

$\hat{A}_\mu^\dagger(x) = \hat{A}_\mu(x) \rightarrow$ hermitian

$\langle \dots A(p, h) \dots | \hat{A}(x) = \epsilon_p^\mu(p)^* e^{+ip \cdot x}$

$\hat{A}^\mu(x) \hat{A}^\nu(x) = \Delta_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} i \frac{-\eta^{\mu\nu} + p^\mu p^\nu / m^2}{p^2 - m^2 + i\epsilon}$

feynman propagator

$\langle f | \hat{S} | i \rangle = \langle f | i \rangle - i \int d^4x \frac{\lambda}{8} \langle f | T(A^\mu A_\mu)^2 | i \rangle$

Feynman Rules:

a rescaling of what λ means Scalar case

$\text{helicity, momentum} \rightarrow \epsilon_h^\mu(p) \text{ (initial)} \left[\frac{\text{wavy line}}{p} = 1 \right]$

$\text{helicity, momentum} \rightarrow \epsilon_h^\mu(p)^* \text{ (final)}$

$\text{Feynman diagram} = -i\lambda (\eta_{\mu\nu} \eta_{\rho\sigma} + \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho})$

$\dots \lambda (\eta_{\mu\nu} \hat{A}^\mu A^\nu) (\eta_{\rho\sigma} \hat{A}^\rho \hat{A}^\sigma)$

Sum over all Contractions.
We want to put in a factor that accounts for all the symmetric contractions

for the space indy of field

$$M_{\mu\nu} = -i \frac{\eta^{\mu\nu} - p^\mu p^\nu / m^2}{p^2 - m^2 + i\epsilon}$$

Symmetry Factor: $\Sigma = \left(\frac{1}{8}\right)^V \cdot C$

C : # of contractions that gives the same diagram/math expression