

QFT 230 B Lecture 3

10/1/2025

Last time
interpolating
field

* Spin 0 $\langle \underbrace{\Omega}_{\text{vacuum}} | \hat{\phi}(x) | \underbrace{\phi(p)}_{\phi \text{ particles}} \rangle \neq 0$

$\Rightarrow \hat{\phi}(x) | \Omega \rangle$ has same quantum numbers as $|\phi(p)\rangle$

Poincaré Invariance

$$\langle \Omega | \hat{\phi}(x) | \phi(p) \rangle = \sqrt{Z_\phi} e^{-ip \cdot x}$$

Free field

$$\hat{\phi}(x) = \int (dp) [\hat{a}(p) e^{-ip \cdot x} + \text{h.c.}] , \quad Z=1 \text{ for free field, } 0 < Z_\phi < 1 \text{ interacting field}$$

Spin 1

$$\langle \Omega | \hat{A}^\mu(x) | A(p, s) \rangle \neq 0? \quad s=0, \pm 1$$

Is this a good interpolating field? Check Poincaré Invariance

$$= \langle \Omega | \hat{A}^\mu(0) | A(p, s) \rangle e^{-ip \cdot x}$$

$$\stackrel{\text{def}}{=} \underbrace{\int \frac{d^3p}{(2\pi)^3} \sqrt{Z_A} \epsilon_s^\mu(p)}_{\text{vector wave functions}}$$

Free Vector Field:

Claim: $\hat{A}^\mu(x) = \int (dp) \sum_s [\underbrace{\hat{a}_s(p)}_{\text{annihilation operator needs index } s} \epsilon_s^\mu(p) e^{-ip \cdot x} + \text{h.c.}]$

annihilation operator needs index s

$$\hat{a}_s^\dagger(p) | 0 \rangle = | A(p, s) \rangle$$

$$\Rightarrow \langle 0 | \hat{A}^\mu(x) | A(p, s) \rangle = \underbrace{\epsilon_s^\mu(p)}_{\neq 0?} e^{-ip \cdot x} \quad (Z_A=1)$$

$\neq 0?$ Is it compatible w/ Lorentz Invariance

$$\hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(\Lambda) = \Lambda^\mu_\nu \hat{A}^\nu(\Lambda^{-1}x) \quad (\text{supposed to happen})$$

$$\hat{A}^\mu(0) = (\Lambda^{-1})^\mu_\nu \hat{U}^\dagger(\Lambda) \hat{A}^\nu(0) \hat{U}(\Lambda)$$

$$\epsilon_s^\mu(p) = (\Lambda^{-1})^\mu_\nu \underbrace{\langle 0 | \hat{U}(\Lambda)^\dagger \hat{A}^\nu(0) \hat{U}(\Lambda) | A(p, s) \rangle}_{\neq 0}$$

$$= (\Lambda^{-1})^\mu_\nu \epsilon_{s'}^\nu(\Lambda p) \underbrace{\langle 0 | \hat{U}(\Lambda)^\dagger \hat{A}^\nu(0) \hat{U}(\Lambda) | A(\Lambda p, s') \rangle}_{= | A(\Lambda p, s') \rangle} D_{s's}^{(1)}(W(\Lambda, p))$$

$$\boxed{\Lambda^\mu_\nu \epsilon_s^\nu(p) = \epsilon_{s'}^\mu(\Lambda p) D_{s's}^{(1)}(W(\Lambda, p))} \quad \text{what does this mean?}$$

$$\Lambda^\mu_\nu \epsilon^\nu_s(p) = \epsilon^\mu_{s'}(\Lambda p) D^{(1)}_{s's}(W(\Lambda, p)) \quad \text{--- Eq. 2}$$

$$\underbrace{\hat{U}(\Lambda)}_{\text{matrix}} \underbrace{|A(p, s)\rangle}_{\text{vector}} = \underbrace{|A(\Lambda p, s')\rangle}_{\text{same vector with } \Lambda p \text{ at } s'} D^{(1)}_{s's}(W(\Lambda, p))$$

Question: Can we find/construct $\epsilon^\mu_s(p)$ that satisfies this?
Yes. (But no for $m \rightarrow 0$)

IDEA: Start w/ reference momentum

$$k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

analogous to:

first construct

$$[|k, s\rangle]$$

① construct $\epsilon^\mu_s(k)$

② $\epsilon^\mu_s(p)$ from boosting

① $\Lambda = R \in SO(3)$ set $\mu=1$

$$R^i_j \epsilon^j_s(k) = \epsilon^i_{s'}(k) \begin{pmatrix} e^{-i\vec{\theta} \cdot \vec{J}_{(1)}} \end{pmatrix}_{s's}$$

$$= \begin{pmatrix} e^{-i\vec{\theta} \cdot \vec{J}} \end{pmatrix}_{ij}$$

$$J_{(1)}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_{(1)}^1 = i \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$J_{(1)}^2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$J^3 = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

set $\mu=0$: $\Lambda^\mu_\nu \epsilon^\nu_s(p) = 0 = \text{LHS}$

$$0 = \epsilon^0_{s'}(k) D^{(1)}_{s's}(R) \neq 0$$

$$\Rightarrow \boxed{\epsilon^0_{s'}(k) = 0} \quad \forall s$$

$$\epsilon^\mu_s(k) = \begin{pmatrix} 0 \\ \vec{e}_s \end{pmatrix} \quad \vec{e}_s = ?$$

$$\boxed{(J^i)^j_k e_s^k = (J^i)_{s's} e_s^j}$$

$$\# \text{ eq} = 3 \cdot 3 \cdot 3 = 27$$

$$\# \text{ unknowns} = 9 \cdot 3 = 27 \quad \leftarrow (e_s^j) = 3 \cdot 3 = 9$$

$$i = 1, 2, 3$$

$$s = 1, 2, 3$$

$$j = 1, 2, 3$$

$\exists$ soln, unique up to rescaling of e
(only direction of e matters)

$$U(\Lambda) U(\Lambda') = U(\Lambda \Lambda') D^{(s)}(\Lambda^{-1} \Lambda')$$

$$U(\Lambda) \epsilon^\mu_s(k) = \epsilon^\mu_{s'}(\Lambda k) D^{(s)}_{s's}(\Lambda)$$

$$\epsilon^\mu_{s=\pm 1}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp 1 \\ -i \\ 0 \end{pmatrix} \quad \epsilon^\mu_{s=0}(k) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{allowed to rescale these}$$

② $\epsilon^\mu_s(p) \stackrel{\text{def}}{=} L(p)^\mu_\nu \epsilon^\nu_s(k)$ claim this is the solution to Eq. ①

$p = L(p) \cdot k$
 $|p, s\rangle = U(L(p)) |k, s\rangle$ indices arbitrary for particle not at rest

$$\begin{aligned} \Lambda^\mu \epsilon_s(p) &= L(\Lambda p)^\mu_\nu W(\Lambda, p)^\nu_\rho L(p)^\rho_\sigma \epsilon^\sigma_s(k) \\ &= L(\Lambda p)^\mu_\nu W(\Lambda, p)^\nu_\rho L(p)^\rho_\sigma \epsilon^\sigma_s(k) \\ &= L(\Lambda p)^\mu_\nu \epsilon^\nu_{s'}(k) D^{(s)}_{s's}(W(\Lambda, p)) \\ &= \epsilon^\mu_{s'}(\Lambda p) D^{(s)}_{s's}(W(\Lambda, p)) \end{aligned}$$

$\sum_{s'}$

Is the solution unique? $L(p)$ is not unique, but choose $\tilde{L}(p)$, where s index becomes helicity $=h$

little group ambiguity

$\tilde{L}(p) = \text{boost along } \vec{p}$
 $\tilde{L}(p) = \text{boost along } x^3, \text{ followed by rotation}$

$\hat{h}(p) |p, s\rangle = \frac{\vec{p} \cdot \vec{J}}{|\vec{p}|} |p, s\rangle$
 component of spin along $\vec{p}$

different choice L gives different meanings to spin index s . not defined for $\vec{p}=0$ (issue for general s)
 (LHC hardly ever measures spin) (not true)

$\Rightarrow \epsilon^\mu_{h=0}(p) = \frac{1}{m} \begin{pmatrix} |\vec{p}| \\ p^0 \frac{\vec{p}}{|\vec{p}|} \\ 0 \end{pmatrix}$

$\epsilon^\mu_{h=\pm 1}(p) = \begin{pmatrix} 0 \\ \mp \cos\theta \cos\phi + i \sin\phi \\ * \\ * \end{pmatrix}$ euler angles

$\frac{\vec{p}}{|\vec{p}|} = \hat{p}$

only 3 polarizations and $\Rightarrow$ 3 independent solutions
 4 vectors
 general: s
 specific: h

we want to write down QFT for Z, ρ bosons, ...

Free fields: $\hat{A}^\mu(x) = \int (dp) \sum_{s=0, \pm 1} [\hat{a}_s(p) \epsilon^\mu_s(p) e^{-ip \cdot x} + \text{h.c.}]$

Klein Gordon equations

$\square \hat{A}^\mu(x) = -m^2 \hat{A}^\mu(x) \Rightarrow (\square + m^2) \hat{A}^\mu(x) = 0 \leftarrow 4 \text{ E.M.}$

$\uparrow$
 $p^2 = m^2$

Recall $\epsilon^0_s(k) = 0$
 $k \cdot \epsilon_s(k) = 0$

Additional Constraint $\Rightarrow p \cdot \epsilon_s(p) = 0 \Rightarrow \partial_\mu A^\mu(x) = 0$
 Additional E.o.M.

$$\epsilon_{\mu\nu}^{\lambda\sigma}(k) = \frac{1}{T} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\epsilon_{\mu\nu}^{\lambda\sigma}(k) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

opposite sign of helicity

$$\partial_\mu \hat{A}^\mu(x) = \int (dp) \sum [\hat{a}_s(n)] \left[\epsilon_s^\mu(p) (-ip_\mu) e^{-ip \cdot x} + h.c. \right] = 0$$

50 ways of
looking at a
black bird

Classical $\hat{A}^\mu(x)$
free

Linear equations of motion
second order in ∂ 's

$$0 = \square A^\mu + M^2 A^\mu + \alpha \partial^\mu \partial_\nu A^\nu \quad \rightarrow \text{more general equation}$$

↑
arbitrary index α

$\swarrow 4 E \cdot M$