

QFT Lecture 2 9/29/2025

QM: Spin S rep of $SO(3)$ ($S = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$)

$$\langle s' | \hat{J}^i | s \rangle = (J^i(s))_{s's}$$

$$J^1(\frac{1}{2}) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J^2(\frac{1}{2}) = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J^3(\frac{1}{2}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$O_{s's}^{(s)}(R) = (e^{-i\vec{\theta} \cdot \vec{J}(s)})_{s's}$$

$$R = \text{fcn of } \vec{\theta}$$

QFT $k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \hat{U}(\Lambda) = ?$

↑
at rest

general momentum $\rightarrow R \in SO(3): \hat{U}(R) |k, s\rangle = |k, s'\rangle O_{s's}^{(s)}(R)$

$$p^\mu = L(p)^\mu_\nu k^\nu$$

$L(p)$ = Lorentz Transformation

$$p^\mu = \begin{pmatrix} \sqrt{p^2 + m^2} = E \\ p \end{pmatrix}$$

$$k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{LT} \begin{pmatrix} E \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix}$$

boosting along z axis so boosted momentum has correct energy

$L(p)$ isn't unique
defined for arbitrary
momentum p

$$p^2 = m^2$$

↓ rot

$$\begin{pmatrix} E \\ \vec{p} \end{pmatrix}$$

Wigner rotation
associated
with P
and Λ

arbitrary transformation standard Lorentz Transformation

$$W(\Lambda, p) \stackrel{\text{def}}{=} L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p) \in SO(3)$$

$$W(\Lambda, p) \cdot k = L(\Lambda p)^{-1} \cdot \Lambda \cdot L(p) \cdot k$$

$\underbrace{\hspace{10em}}_{\Lambda p}$

$$L(p) \cdot k = p$$

$$k = L(p)^{-1} \cdot p$$

Define $\hat{u}(\Lambda) |p, s\rangle \stackrel{\text{def}}{=} \underbrace{\hspace{10em}}_{\Lambda p} = k$

$\uparrow$
 Same spin
 $\stackrel{\text{def}}{=} \sum_{s'} |\Lambda p, s'\rangle \mathcal{D}_{s's}^{(s)}(W(\Lambda, p))$

$$|\Lambda p, s'\rangle \mathcal{D}_{s's}^{(s)}(W(\Lambda, p))$$

Check:

- * rep?
- * unitary?

Representation:

~~rep~~

Assume $\hat{u}(\Lambda) = \text{rep}$

$$\hat{u}(\Lambda) |p, s\rangle = \hat{u}(L(\Lambda p) \cdot W(\Lambda, p) \cdot L(p)^{-1}) \cdot |p, s\rangle$$

$$= \hat{u}(L(\Lambda p)) \cdot \hat{u}(W(\Lambda, p)) \cdot \hat{u}(L(p)^{-1}) \times |p, s\rangle$$

$$|p, s\rangle \stackrel{\text{def}}{=} \hat{u}(L(p)) |k, s\rangle$$

$$= |k, s'\rangle \mathcal{D}_{s's}^{(s)}(W(\Lambda, p))$$

$$= |\Lambda p, s'\rangle \mathcal{D}_{s's}^{(s)}(W(\Lambda, p))$$

$$\langle p', s' | p, s \rangle = \langle p' | p \rangle \delta_{s's}$$

$$= (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \cdot 2E_{\vec{p}}$$

$$\Rightarrow \hat{u}(\Lambda^{-1}) = \hat{u}(\Lambda)^\dagger = \hat{u}(\Lambda)^{-1} \dots$$

Little Group Ambiguity

$$k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

reference momentum

$$w \cdot k = k$$

$w \in$ Little group of $k^\mu = SO(3)$

$$L(p) \cdot k = p$$

entire 3x3 rotation matrix of ambiguity
arbitrary rotation $\in SO(3)$

$$L'(p) \cdot k = p$$

$$L'(p) = L(p) \cdot R$$

...

show
can
you

$$|p, s\rangle' = |p, s'\rangle D_{s's}^{(s)}(R(p))$$

summing over s'

$$S\text{-matrix: } \langle p_{n+1}, s_{n+1}, \dots, p_r, s_r | \hat{S} | p_1, s_1, \dots, p_n, s_n \rangle$$

$$\mapsto \langle \dots | \hat{U}(\Lambda)^\dagger \hat{S} \hat{U}(\Lambda) | \dots \rangle$$

$$\hat{U}(\Lambda) |p, s\rangle$$

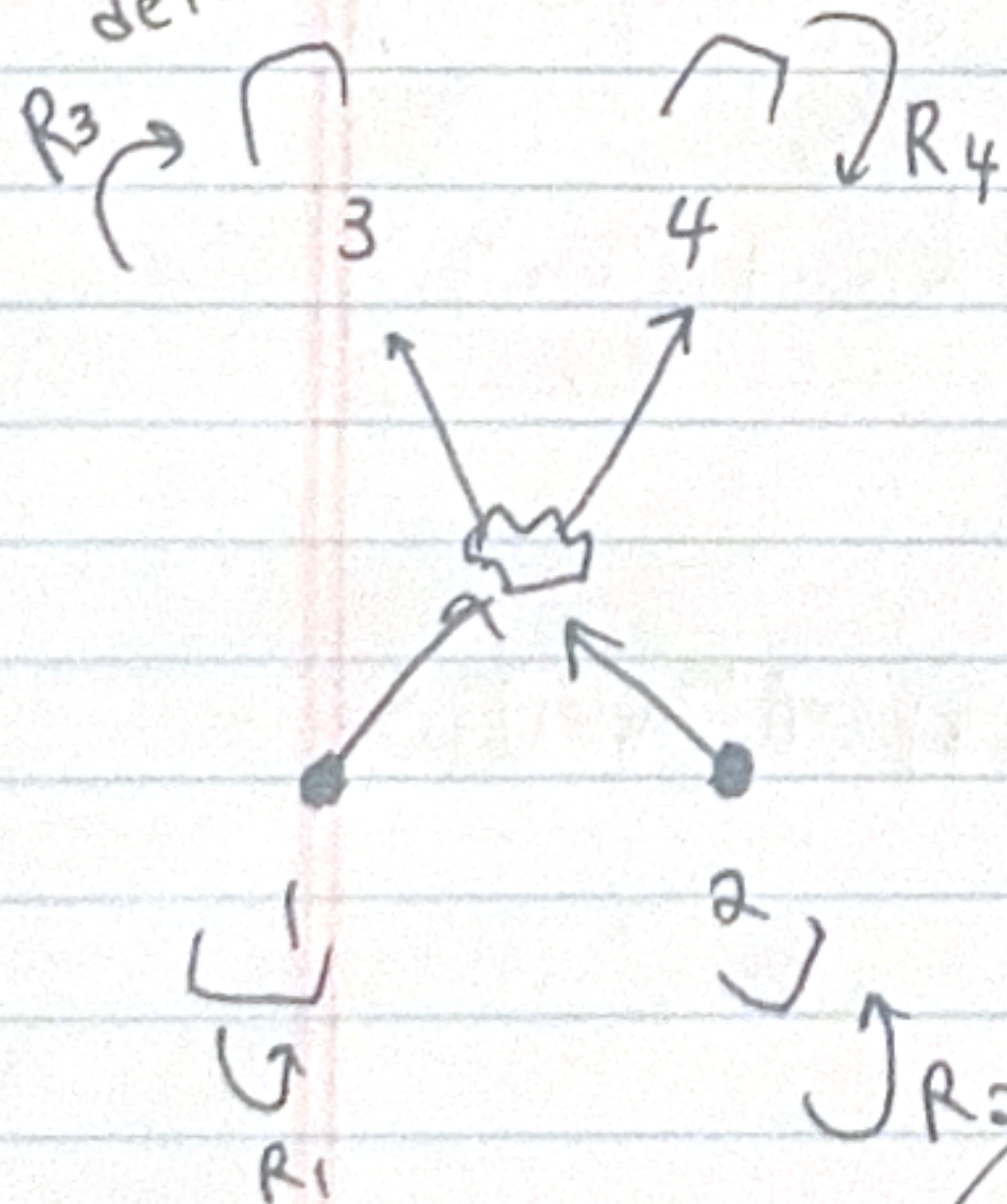
$$= \langle p_{n+1}, s'_{n+1}, \dots | \hat{S} | s' \rangle D_{s's}^{(s)}(R(p_1))$$

$$\times D_{s'_2 s_2}^{(s)}(R(p_2))$$

x ...

change of spin ^{quantization} axis

detectors measuring p, s



helicity

$$|k, s\rangle \quad s = \text{eigenvalue of } (J^3)_{s_z} = J^3$$

$$|p, s\rangle \quad s = \text{eigenvalue of } J^3 \quad (\text{exercise})$$

$$|p, h\rangle \quad h = \frac{\vec{p} \cdot \vec{J}}{|\vec{p}|} \quad h: \text{spin along } \vec{p} \text{ axis}$$

$$s = -s, -s+1, \dots, s-1, s$$

$$h = -s, -s+1, \dots, s-1, s$$

$$|p, h\rangle = \hat{U} |p, s\rangle \dots$$

Spin 1 (massive)

claim: simpler than massless spin 1

1 particle state $|p, s\rangle$ $s = -1, 0, 1$

$$p^2 = m^2 \neq 0$$

$[m \rightarrow 0$ massless spin 1 has 2 polarizations (EM wave)]

discontinuous difference between massless and massive

you can't boost a massive particle to C and turn it to massless particle

spin 0: $\langle \Omega | \hat{\phi}(x) | p \rangle = \sqrt{Z} e^{-ip \cdot x}$ Scalar field
interpolates

$\uparrow$ vacuum state $\uparrow$ no spin 1 particle state

spin 1: $\langle \Omega | \hat{\phi}(x) | p, s \rangle ?$

4 μ but 3 spin states $-1, 0, 1$

$\langle \Omega | \hat{A}^\mu(x) | p, s \rangle ? \checkmark$ $\mu = 0, 1, 2, 3$

$$\langle \Omega | \hat{A}^\mu(x) | p, s \rangle = \underbrace{\langle \Omega | e^{i\hat{p} \cdot x}}_{\langle \Omega |} \hat{A}^\mu(0) \underbrace{e^{-i\hat{p} \cdot x} | p, s \rangle}_{e^{-i\hat{p} \cdot x} | p, s \rangle}$$

$$= e^{-ip \cdot x} \langle \Omega | \hat{A}^\mu(0) | p, s \rangle$$

~~4~~ = 4-vector $\epsilon_s^\mu(p)$

$$\langle \Omega | \hat{A}^\mu(0) | p, s \rangle \xrightarrow{\Lambda} \langle \Omega | \hat{U}(\Lambda)^\dagger \hat{A}^\mu(0) \hat{U}(\Lambda) | p, s \rangle$$

$$= \Lambda^\mu_\nu A^\nu(0)$$

$$\langle \Omega | \hat{A}^\mu(x) | p, s \rangle = \sqrt{Z} \underbrace{\epsilon_s^\mu(p)}_{\text{polarization vector}} e^{-ip \cdot x}$$

$\mu = 0, 1, 2, 3$
 $s = -1, 0, 1$

Free QFT ($Z=1$)

$$\hat{A}^\mu \sim \hat{a} + \hat{a}^\dagger$$

$$\hat{\phi}(x) = \int (dp) [\hat{a}(p) e^{-ip \cdot x} + \hat{a}^\dagger(p) e^{ip \cdot x}]$$

↓
Lorentz
invariant
measure

$$p^2 = m^2$$

creation operators have spin on them

$$\hat{a}^\dagger(p)|0\rangle = |p\rangle$$

$$\hat{a}_s^\dagger(p)|0\rangle = |p, s\rangle$$

$$\hat{A}^\mu(x) = \int (dp) \sum_s [\hat{a}_s(p) \epsilon_s^\mu(p) e^{-ip \cdot x} + \hat{a}_s^\dagger(p) \epsilon_s^\mu(p)^* e^{ip \cdot x}]$$

$$\Rightarrow \langle 0 | \hat{A}^\mu(x) | p, s \rangle = \epsilon_s^\mu(p) e^{-ip \cdot x}$$

imply

$$\int d^3x \delta^3(\vec{x} - \vec{y}) f(\vec{x}) = f(\vec{y})$$

$$\int (dp) \underbrace{\langle R | p \rangle}_{\text{Lorentz invariant delta function}} f(p) = f(k)$$

$$= \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}$$

Lorentz
invariant
delta
function

$$(2\pi)^3 \delta^3(\vec{p} - \vec{k}) \cdot 2E_{\vec{p}}$$

Fock Space

$$\hat{a}_s(p)|0\rangle = 0 \quad \forall p, s$$

$$|p_1, s_1, \dots, p_n, s_n\rangle = \hat{a}_{s_1}^\dagger(p_1) \dots \hat{a}_{s_n}^\dagger(p_n) |0\rangle$$

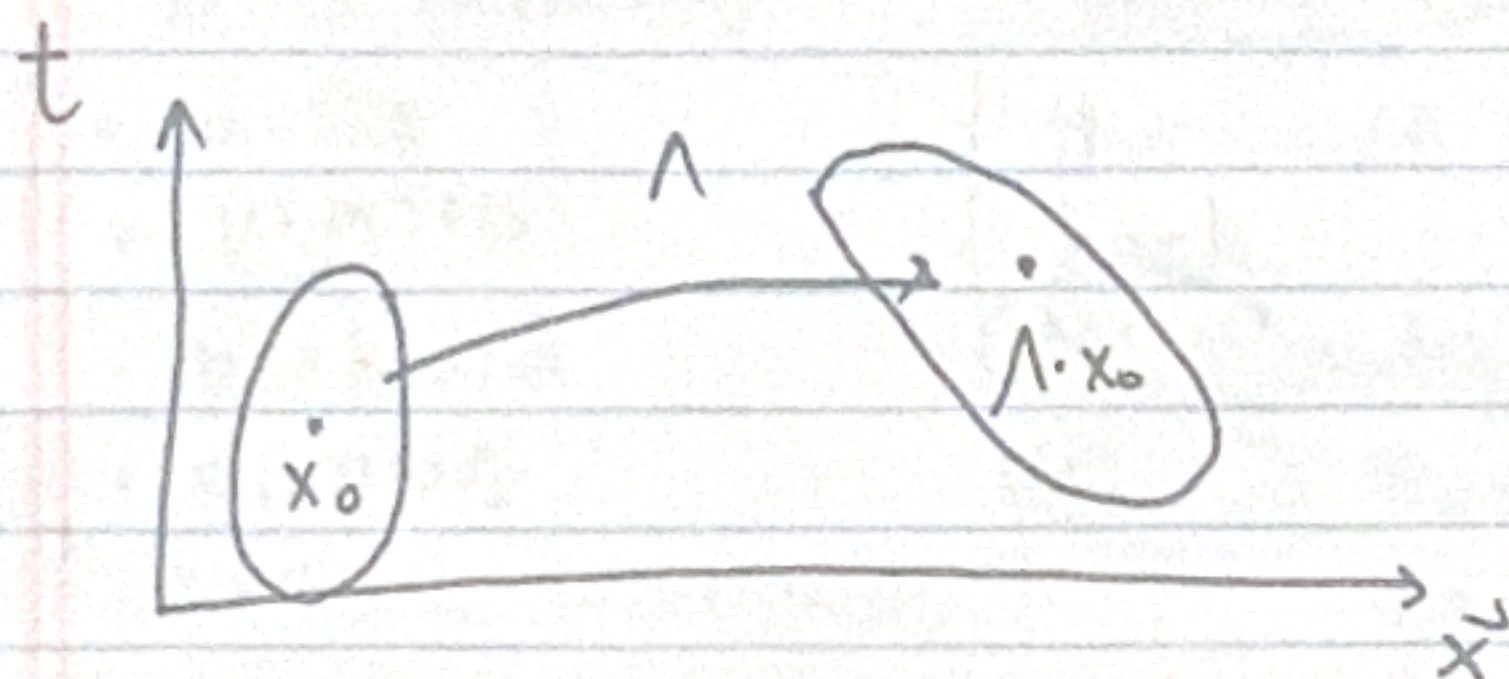
$$[\hat{a}_s(p), \hat{a}_{s'}^\dagger(p')] = \delta_{s's} \underbrace{\langle p' | p \rangle}_{\text{proportional to } \delta(\vec{p}' - \vec{p})}$$

proportional

to $\delta(\vec{p}' - \vec{p})$

Is $\hat{A}^\mu(x)$ a vector field?

Classical: $A^\mu(x) \xrightarrow{\Lambda} A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$



$$\hat{U}(\Lambda) |p, s\rangle = |\Lambda p, s'\rangle D_{s's}(W(\Lambda, p))$$

$$\Rightarrow \hat{U}(\Lambda)^\dagger \hat{A}^\mu(x) \hat{U}(\Lambda) = \Lambda^\mu_\nu \hat{A}^\nu(\Lambda^{-1}x)$$

Unitary transformation on Hilbert space

+
Lorentz transformation on field

is true

Can we choose $\epsilon_s^\mu(p)$ s.t.
choice of polarization vectors.

Spoiler alert: yes.

($m \rightarrow 0$ no.) leads to gauge invariance

linear combo of s' given by Wigner rotation

$$\Leftrightarrow \Lambda^\mu_\nu \epsilon_s^\nu(p) = \epsilon_{s'}^\mu(\Lambda p) D_{s's}(W(\Lambda, p))$$

$$\hat{U}(\Lambda) |p, s\rangle = |\Lambda p, s'\rangle D_{s's}(W(\Lambda, p))$$

" $\epsilon_s^\mu(p)$ transforms like a 1-particle state"

↑
form a representation of the Lorentz group, isomorphic to
transformation of 1 particle state