

- What is (relativistic) QFT?

simple
harmonic
oscillator

$$\hbar \neq 0 \quad c \neq \infty$$

$$E_n = \hbar(\omega + \frac{1}{2}) \quad v \approx c$$

$$E \sim mc^2$$

$$\equiv \updownarrow \hbar\omega$$

low lying states

QFT: few particles

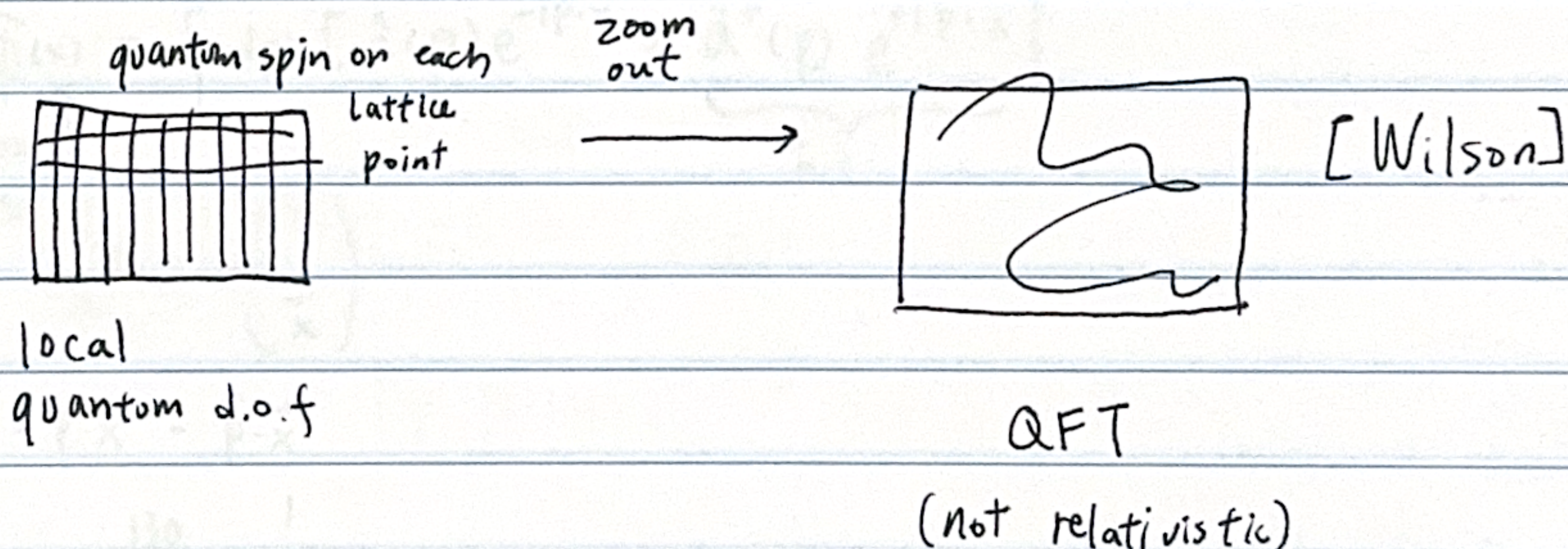
$$\hbar = 1 \quad c = 1$$

quantum theory of EM field

- ① "Quantize" Fields $A_\mu(x) \rightarrow \hat{A}_\mu(x)$ [Dirac]

- ② QM + SR $\rightarrow$ QFT [Steven Weinberg]

- ③ Effective Field Theory



- What is Relativity?

Spacetime Symmetry

rotation invariance

time translation invariance

spatial translation invariance

Lorentz invariance (no preferred rest frame)

flat spacetime

- "Lorentz" Transformations

$$\Lambda^\mu_\nu = 4 \times 4$$

$$\mu, \nu = 0, 1, 2, 3$$

$$\Lambda \eta \Lambda^T = \eta$$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \eta^{\mu\nu}, \eta_{\mu\nu}$$

$$* \det(\Lambda) = 1$$

$$|\Lambda^0_i| \geq 1 \quad i = 1, 2, 3$$

ensure Lorentz group consists just rotation and boosts but not
parity and time reversal invariance

violated in nature

time reversal: run movie backwards and not have a piece of paper going up

- Free Scalar Fields

$$\hat{\phi}(x) = \int (dp) \left[\hat{a}(p) e^{-ip \cdot x} + \underbrace{\hat{a}^\dagger(p) e^{+ip \cdot x}}_{\text{h.c.}} \right]$$

$$x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

$$p \cdot x = p^0 x^0 - \vec{p} \cdot \vec{x}$$

$$(dp)_m = \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}$$

mass of
particle

$$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

$$\hat{a}(p)|0\rangle = 0 \quad \forall p$$

$$\hat{a}^\dagger(p_1) \dots \hat{a}^\dagger(p_n)|0\rangle = |p_1, \dots, p_n\rangle$$

Peskin
Shroder
Convention
Notation

relativistically normalized

$$\langle k | p \rangle = 2 E_p (2\pi)^3 \delta^3(\vec{k} - \vec{p}) \text{ boost invariant}$$

$$\langle \Lambda k | \Lambda p \rangle = \langle k | p \rangle$$

- Relationship btw Particles $\longleftrightarrow$ Fields (interacting)
[heading towards spin 1]

$$\text{Scalar Fields : } \hat{\phi}(x) \longleftrightarrow |p\rangle$$

Poincaré Group

exact $\downarrow$ interpolating field

$$\langle \Omega | \hat{\phi}(x) | p \rangle = \sqrt{Z} e^{-ip \cdot x} \text{ follows from Lorentz invariance}$$

$\uparrow$
exact ground state

Poincaré Invariance

$$\hat{H} |\Omega\rangle = 0$$

operator $\downarrow$ state with eigenvalue q
 $\hat{q} |q\rangle = q |q\rangle$
 $\uparrow$ eigenvalue

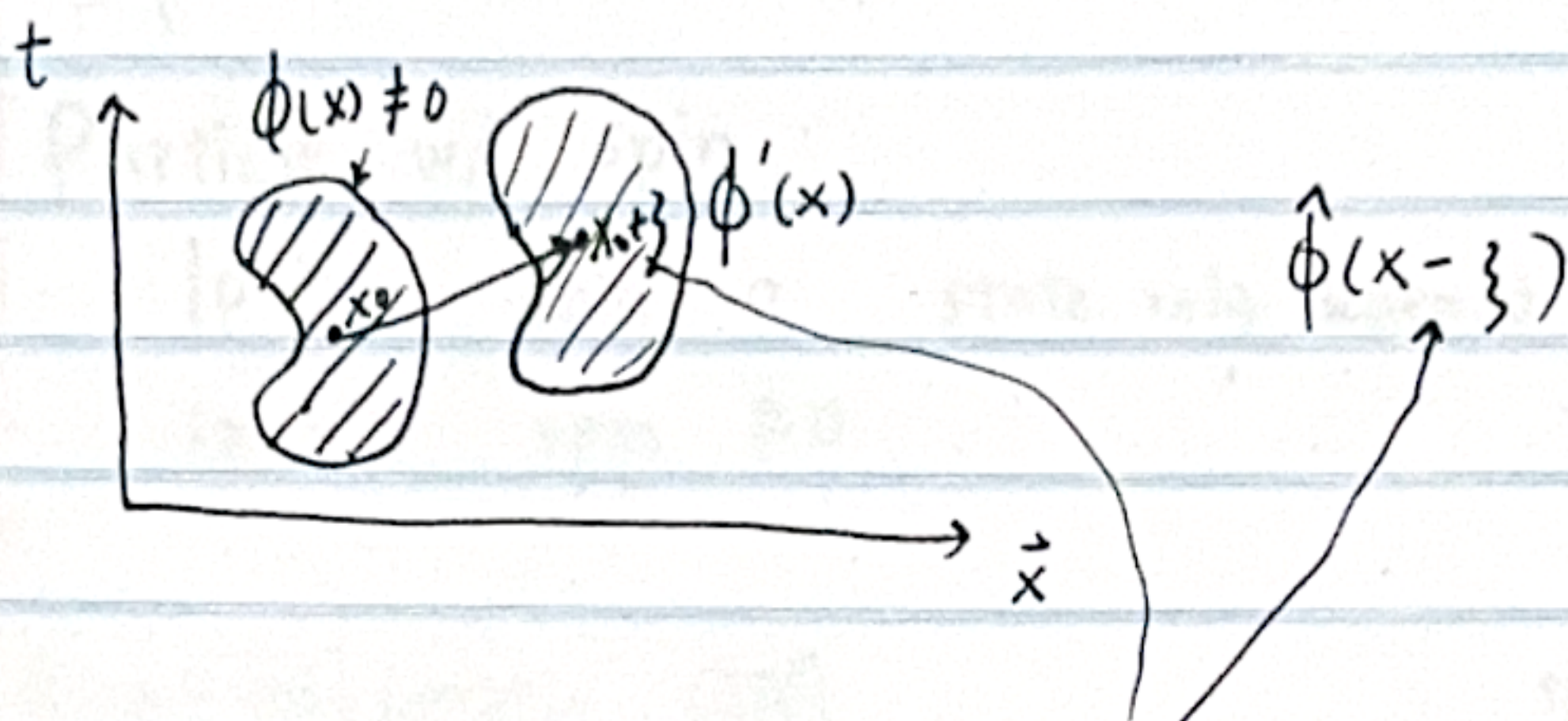
Poincaré Invariance : $\exists$ a unitary representation of Poincaré group act on states

$$\hat{U}(\Lambda) |p\rangle = |\Lambda p\rangle \text{ boost}$$

$$e^{-i\vec{z} \cdot \hat{\vec{p}}} |p\rangle = e^{+i\vec{p} \cdot \vec{z}} |p\rangle$$

$$\hat{p}^\mu |p\rangle = p^\mu |p\rangle$$

$$\begin{aligned} & \langle \Omega | \hat{\phi}(x) | p \rangle \\ & \xrightarrow{\vec{z}} \langle \Omega | e^{-i\vec{z} \cdot \hat{\vec{p}}} \hat{\phi}(x) e^{+i\vec{z} \cdot \hat{\vec{p}}} | p \rangle \quad \text{unitary transformation acting on field} \\ & \underbrace{\hspace{10em}}_{\text{spacetime translation}} = \hat{\phi}(x - \vec{z}) \end{aligned}$$



$$\hat{U}(\Lambda)^\dagger \hat{\phi}(x) \hat{U}(\Lambda) = \hat{\phi}(\Lambda^{-1}x)$$

choose $z = x$

$$\hat{\phi}(x) = e^{ix \cdot \hat{p}} \hat{\phi}(0) e^{-ix \cdot \hat{p}}$$

$$\langle \Omega | \hat{\phi}(x) | p \rangle = \langle \Omega | e^{i\hat{p} \cdot x} \hat{\phi}(0) e^{-i\hat{p} \cdot x} | p \rangle$$

$$= \langle \Omega |$$

$$= e^{-ip \cdot x} \langle \Omega | \hat{\phi}(0) | p \rangle = e^{-ip \cdot x} \langle \Omega | \hat{\phi}(0) | \Lambda p \rangle$$

doesn't depend on anything

$$\hat{\phi}(0) = \hat{U}(\Lambda)^\dagger \hat{\phi}(0) \hat{U}(\Lambda)$$

you have to choose some origin $x=0$

Lorentz invariant

$$\Lambda \cdot 0 = 0$$

↑
constant

$$0 < \sqrt{z} < 1$$

$$p^2 = m^2$$

$$\sim p^\mu p_\mu$$

$$f(p) = f(\Lambda p)$$

$$\langle \Omega | \hat{\phi}(x) | p \rangle = \sqrt{z} e^{-ip \cdot x}$$

→ $\langle 0 |$

$z \rightarrow 1$

free:

"Interpolating Field" connects vacuum state with 1 particle state

Spin: how it transforms under Lorentz transformation

Particle w/ spin:

$|p\rangle$ spin 0 state only depends on the 4-momentum

$|p, s\rangle$ spin $\neq 0$

transform
under
rotation
group

Non-relativistic QM

$|s\rangle \xrightarrow{R \in \cancel{SO(3)} \cancel{SU(2)} SO(3)}$

$$\sum_{s'=-s}^{s'} D_{s's}^{(s)}(R) |s'\rangle$$

spin labels which representation of rotation group

$$D(R_1)D(R_2) = D(R_1 R_2)$$

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

"m"

$|s, m\rangle$
s

D's
are
SU(2)
matrices

$$D^{(0)} = \mathbb{I}$$

spin label on a state is that state transforms nontrivial way under rotations.

Lorentz Transformations

$$|p\rangle \mapsto \hat{U}(\Lambda) |p\rangle = |\Lambda p\rangle \quad \text{unitary rep of Lorentz Group}$$

$$|p, s\rangle \mapsto \hat{U}(\Lambda p) |p\rangle \stackrel{?}{=} |\Lambda p, s\rangle \quad \text{because } \Lambda \text{ has rotations}$$

Wigner Representation - "Induced" reps

$$\text{Idea: } p^\mu \rightarrow k^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

particle
at
rest

Pauli matrices

$$\Lambda = R$$

rotation

spin $\frac{1}{2}$:

QM:

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

$$\xrightarrow{R}$$

$$e^{-i\vec{\theta} \cdot \vec{\sigma} / 2} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

unitary matrix

tells me which rotations I'm doing

$|\vec{\theta}| = \text{rot angle (right handed)}$

$\vec{\theta}$ points along rot axis

Continued:
spin $\frac{1}{2}$

$$D_{s's}^{(\frac{1}{2})}(\vec{\theta}) = \left(e^{-i\vec{\theta} \cdot \vec{\sigma}} \right)_{s's} \quad \text{consistent}$$

$$D_{s's} \neq D_{ss'}$$

$$|k, s\rangle \xrightarrow{\vec{\theta}} |k, s'\rangle D_{s's}^{(\frac{1}{2})}(\vec{\theta})$$

$\xrightarrow{\quad \quad \quad} \sum_{s'}$

What happens
in not
finite dimensional?

What are the possible unitary representations of the rotation group
^
finite-dimensional

k^μ = "standard 4-momentum" particle at rest

~~Left~~ "Little Group": W^μ = Lorentz Transformations that preserve k
 $W^\mu, k^\nu = k^\nu$ subgroup of Lorentz group that keeps k^ν invariant

special kind
of Lorentz
transformation

$$\hat{U}(\Lambda) |p, s\rangle = ?$$

$$\text{s.t. } \hat{U}(W) |k, s\rangle = |k, s'\rangle D_{s's}^{(\frac{1}{2})}(W)$$