

Unveiling the Fascinating World of p – *adic* Numbers

Jenny Guanni Qu

University of California, Davis

May 31, 2023



Textbook Used

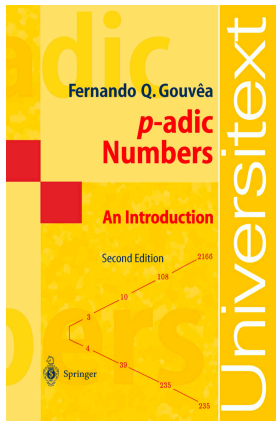


Figure: P-adic numbers

Basics of p – Adic Numbers

Any integer can be expressed uniquely as ± 1 times a product of primes.

Given a positive integer m and a prime p , we can write it “in base p ,” that is, in the form

$$m = a_0 + a_1p + a_2p^2 + \cdots + a_np^n = \sum_{i=0}^n a_i p^i$$

with $a_i \in \mathbb{Z}$ and $0 \leq a_i \leq p - 1$.

Example

Take $p=3$, and consider the number $\frac{24}{17}$, then we have

$$24 = 0 + 2 \times 3 + 2 \times 3^2 = 2p + 2p^2$$

$$17 = 2 + 2 \times 3 + 1 \times 3^2 = 2 + 2p + p^2$$

$$\frac{24}{17} = \frac{2p + 2p^2}{2 + 2p + p^2} = p + p^3 + 2p^5 + p^7 + \dots$$

One Observation

We observe: $-1 = (p-1) + (p-1)p + (p-1)p^2 + (p-1)p^3 + \dots$

$$-1 + 1 = 0 = (p-1+1) + (p-1)p + (p-1)p^2 + (p-1)p^3 + \dots$$

$$= p + (p-1)p + (p-1)p^2 + (p-1)p^3 + \dots$$

$$= 0 + p^2 + (p-1)p^2 + (p-1)p^3 + \dots$$

$$= 0 + 0 + p^3 + (p-1)p^3 + \dots$$

If we add 1, we get 0.

e.g., if $p=3$, $-1 = 222222222222\dots$

The p – adic Valuation

For each integer $n \in \mathbb{Z}, n \neq 0$, let $v_p(n)$ be the unique positive integer satisfying

$$n = p^{v_p(n)} n' \quad \text{with} \quad p \nmid n'$$

We extend v_p to the field of rational numbers as follows: if $x = \frac{a}{b} \in \mathbb{Q}^\times$, then

$$v_p(x) = v_p(a) - v_p(b)$$

$$v_p(0) = +\infty$$

For all x and $y \in \mathbb{Q}$, we have

- i) $v_p(xy) = v_p(x) + v_p(y)$
- ii) $v_p(x + y) \geq \min \{v_p(x), v_p(y)\}$.

Example

Take $p = 2$

$$v_2(20) = v_2(2^2 \times 5) = 2$$

$$v_2(2) = v_2(2^1) = 1$$

$$v_2(150) = v_2((30)(5)) = v_2(30) + v_2(5) = v_2(2^1 \times 15) + 0 = 1 + 0 = 1$$

p -adic Absolute Value

Definition

Definition 2.1.4 For any $x \in \mathbb{Q}$, we define the p -adic absolute value of x by

$$|x|_p = p^{-v_p(x)}$$

if $x \neq 0$, and we set $|0|_p = 0$.

Example

$$|4|_2 = \frac{1}{2^2} = \frac{1}{4}$$

$$|24|_3 = \frac{1}{3^1} = \frac{1}{3}$$

$$|23|_2 = \frac{1}{2^0} = 1 = |333|_2$$

$$|2^{10000000}|_2 = \frac{1}{2^{10000000}}$$

P-adic Metric Space

The absolute value on the field $\mathbb{k}$ is a function

$$|\cdot| : \mathbb{k} \longrightarrow \mathbb{R}_+$$

that satisfies the following conditions:

- i) $|x| = 0$ if and only if $x = 0$
 - ii) $|xy| = |x||y|$ for all $x, y \in \mathbb{k}$
 - iii) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{k}$.
 - iv) Non-archimedean property: $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{k}$;
- P-adic numbers satisfy the non-archimedean property.

$$d(x, y) = |x - y|$$

Metric Space with Non-Archimedean Property: Ultrametric Space.

$$d(x, y) = |x - y| = |x - z + z - y| = |(x - z) + (z - y)| \leq \max\{d(x, z), d(z, y)\}$$

All "triangles" are isosceles

In this Ultrametric Space, all triangles are isosceles.

Proof.

Let x, y and z be three elements of our space (the vertices of our "triangle"). The lengths of the sides of the "triangle" are the three distances

$$d(x, y) = |x - y|$$

$$d(y, z) = |y - z|$$

$$d(x, z) = |x - z|$$

WLOG, let $|x - z|$ be the longest side.

$$x - z = (x - y) + (y - z) = |(x - y) + (y - z)| \leq \max\{|(x - y) + (y - z)|\}$$

Then $|x - z|$ is equal to the bigger of the two. In any case, two of the "sides" are equal. □

Cauchy Sequence

Theorem

Every p -adic integer has p -adic norm ≤ 1

Consider the sequence where each $0 \leq p_i \leq p - 1$

$$a_0 = b_0$$

$$a_1 = b_0 + b_1p$$

$$a_n = b_0 + b_1p + b_2p^2 + \cdots + b_np^n$$

$$|a_m - a_n|_p = |b_mp^m + b_{m-1}p^{m-1} + \cdots + b_{n+1}p^{n+1}|$$

$$= |p^{n+1}||b_mp^{m-n-1} + \cdots + b_{n+1}|$$

$$|p^{n+1}||b_mp^{m-n-1} + \cdots + b_{n+1}| \leq |p^{n+1}|$$

$$(\text{non-archimedean property}) = \frac{1}{p^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Completion of the Rationals under the p -adic metric

Every p -adic integer can be written in the form:

$$x = b_0 + b_1p + b_2p^2 + \cdots + b_np^n + \cdots$$

with $0 \leq b_i \leq p - 1$, and this representation is unique.

Summary

p -adic Absolute Value $\rightarrow$ Weird notion of distance
 $\rightarrow$ Ultrametric Space $\rightarrow$ Cauchy Sequence
 $\rightarrow$ Completion of The Rationals

Convergence

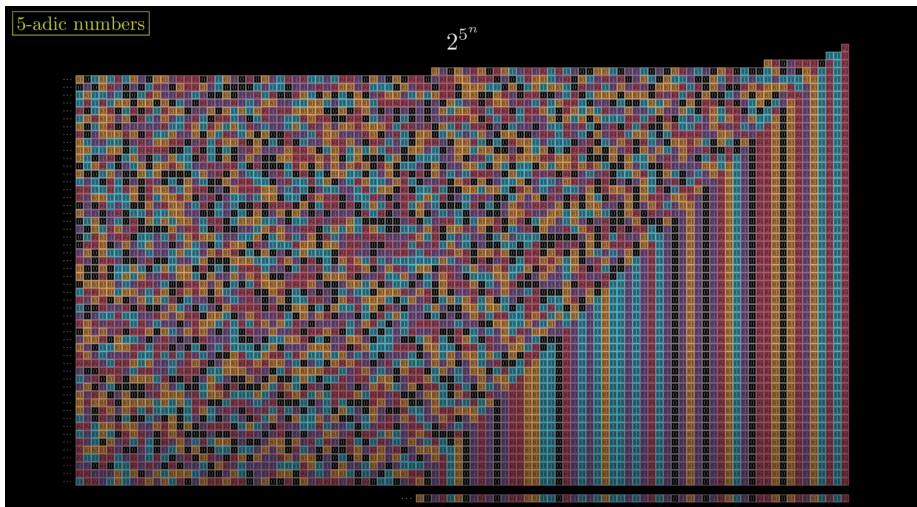


Figure: Convergence from Eric Rowland

P – *adic* Number System

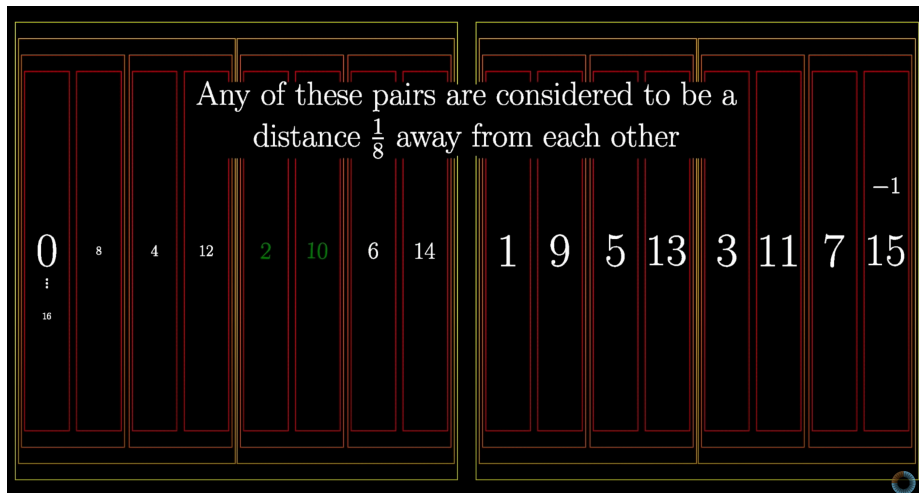


Figure: 3 Blue 1 Brown

Thank you! Any questions?