

The Banach-Tarski Paradox

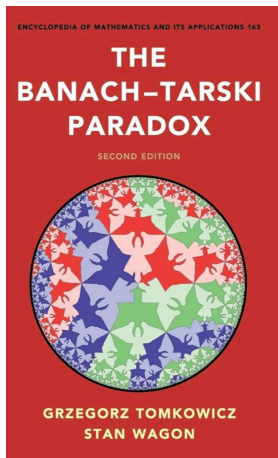
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March 1, 2024



Textbook Used



Motivation of the Banach-Tarski Paradox

Can you take an object and separate it into finitely-many different pieces, and simply re-arrange them (without stretching nor changing the density) into two exact copies of the original item of the same density and size?

G-Equidecomposable

Definition

Definition 3.4. Suppose G acts on X and $A, B \subseteq X$. Then A and B are G equidecomposable (sometimes called finitely G -equidecomposable or piecewise G -congruent) if A and B can each be partitioned into the same finite number of respectively G -congruent pieces. Formally,

$$A = \bigcup_{i=1}^n A_i, \quad B = \bigcup_{i=1}^n B_i,$$

$A_i \cap A_j = \emptyset = B_i \cap B_j$ if $i < j \leq n$, and there are $g_1, \dots, g_n \in G$ such that, for each $i \leq n$, $g_i(A_i) = B_i$.

Remark

You take the set A and cut it into finitely many pieces, and then move the pieces around to form B . Here, you take A_i and use g_i to move them around.

Example

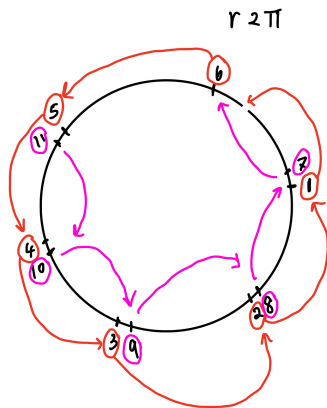


Figure: Spinning points on the circle away

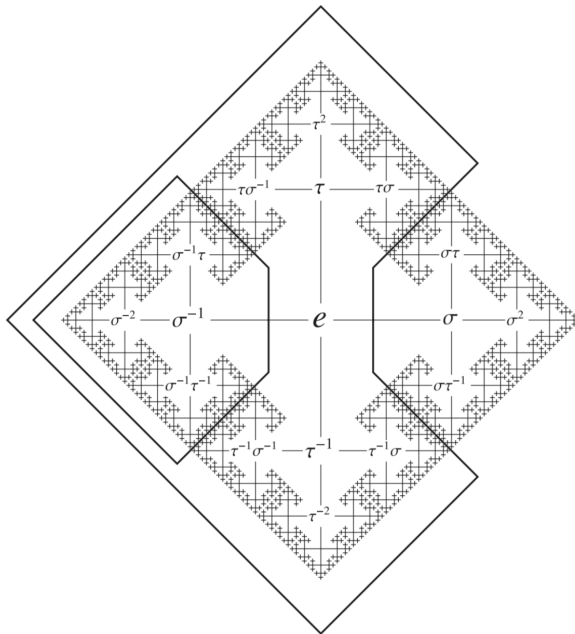
Note: rotation by an irrational multiple of 2π

Free Group on Two Generators

$$F_2 - e = W(\tau) \cup W(\tau^{-1}) \cup W(\sigma) \cup W(\sigma^{-1})$$

$$W(\tau) \cup \tau W(\tau^{-1}) = F_2$$

$$W(\sigma) \cup \sigma W(\sigma^{-1}) = F_2$$



Decomposition

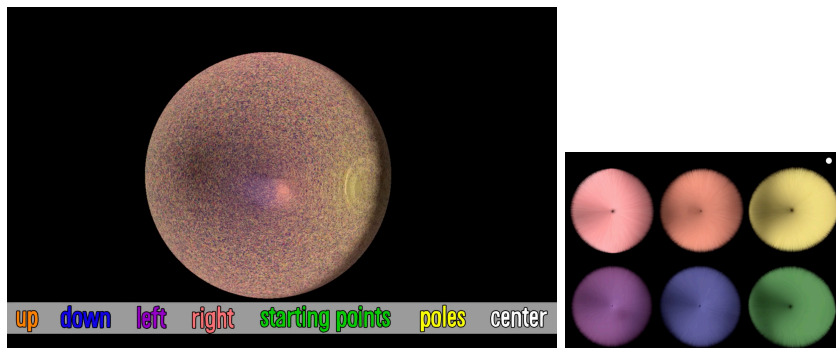


Figure: VSauce

Apply Isometry to certain decomposed parts

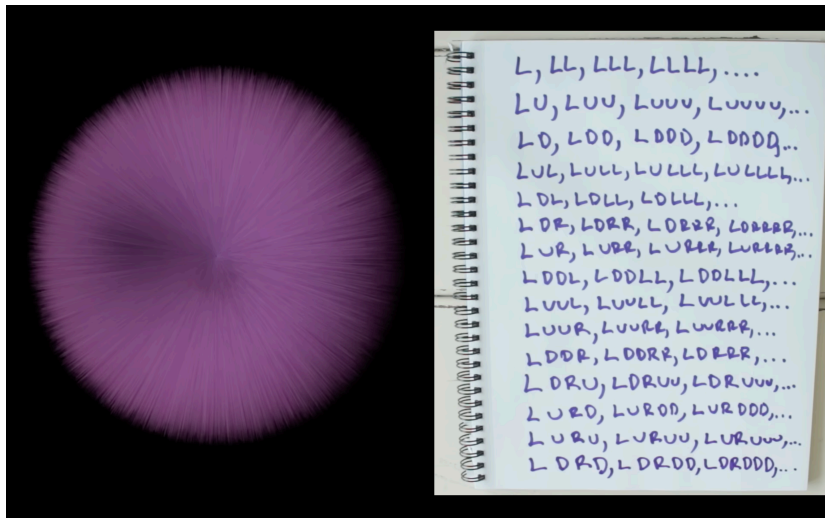


Figure: VSauce

Apply Isometry to certain decomposed parts

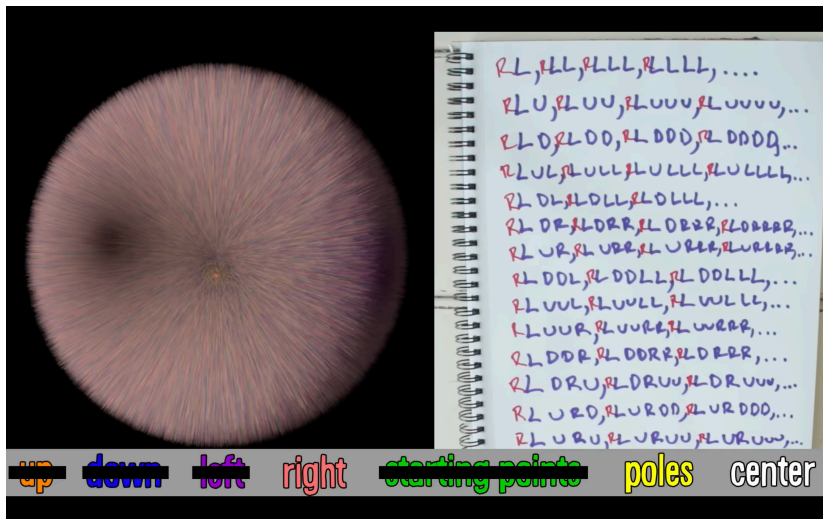


Figure: VSauce

Apply Isometry to certain decomposed parts

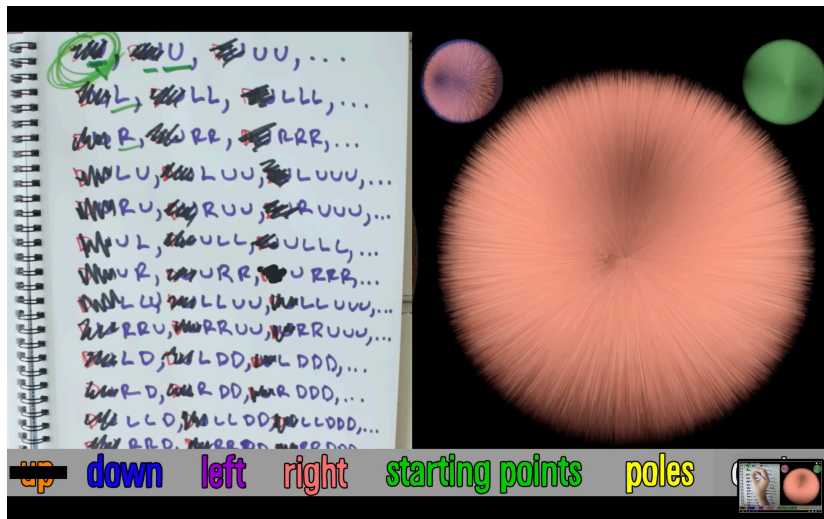


Figure: VSauce

Points inside of the sphere

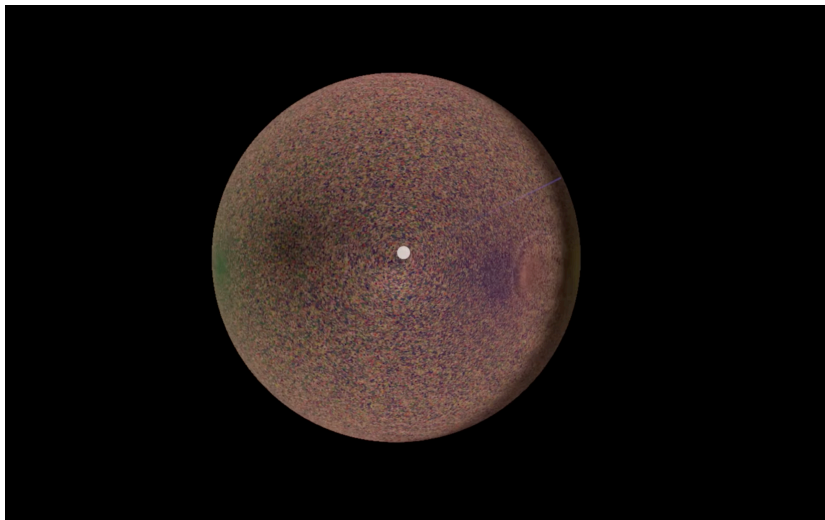


Figure: VSauce

Spinning poles, center, and starting points away and back

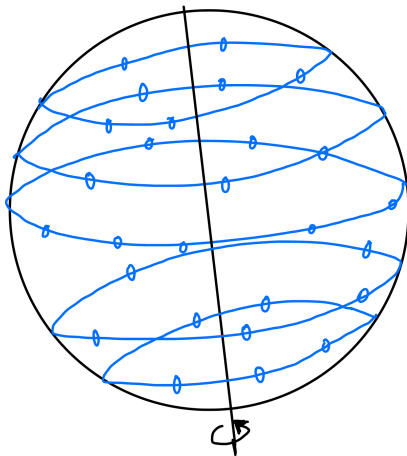


Figure: VSauce

Summary

Two rotations A, B on S^2 that generates F_2

→ Spin away all the points that are fixed by some word of these rotations

→ Use Axiom of Choice to pick 1 element from each orbit of each point

→ Put them in a set K

→ The whole sphere can write as $F_2(K)$

→ Since we can split F_2 into two copies of itself

Apply F_2 to K to get two copies of spheres with points missing.

→ Spin the fixed points back

The Banach Tarski Paradox

Theorem

Theorem 3.12 (Banach-Tarski Paradox, Strong Form) If A and B are any two bounded subsets of $\mathbb{R}^3$, each having nonempty interior, then A and B are equidecomposable.

Theorem

Theorem 3.14. Any tetrahedron in $\mathbb{R}^3$ is equidecomposable with a cube using isometries and pieces that are Lebesgue measurable.

Thank you! Any questions?