

A Tourist's Guide to Algebraic Topology

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1 Introduction / Motivation

The goal of Algebraic Topology is to capture the intrinsic properties of topological spaces by imposing algebraic structure on the space.

In a sense, we want to know when two topological spaces can be treated the same and are topologically indistinguishable.

To do this, in the late 19th and early 20th centuries, mathematicians came up with three main attacks on this idea: The Fundamental group, Homology groups, and Cohomology groups. Here we will discuss the essence of Homology and Cohomology through two key examples of each and show that as functors from the category of chain complexes on a topological space and the category of abelian groups these constructions are isomorphic.

A reader of this paper should be familiar with the theory of groups, rings, and modules; understanding quotient spaces is essential to algebraic topology. We also assume that the reader has the basic notion of a topology on a set and is comfortable with notions of elementary topological spaces such as S^n , D^n , $\mathbb{R}^n$.

To build intuition and a solid foundation we start by exploring the concept of homotopy equivalence, which is an equivalence relation on topological spaces which distinguishes topological spaces depending on whether or not they can be continuously transformed into one another.

We will study invariants of homotopy equivalence, starting with the Fundamental group. The Fundamental group gives us a way to count the number of ways to draw non-contractible loops in a topological space. We will use homotopy theory to study special ways to compute isomorphisms of this group, and we will look at special cases when the topological space has a special kind of structure.

We then explore Simplicial and Singular Homology Theories. Simplicial Homology gives us a natural way to calculate Homology groups and Singular Homology allows us to handle algebraic proofs. Although two homologies are different in construction, they are equivalent at the end. So, one has a choice of methods when calculating homology groups.

Finally, we explore the de Rahm and Cech Cohomology theories and omit the general definition of Sheaf cohomology for the sake of simplicity. De Rahm cohomology allows us to understand the topology of a differential manifold through the language of differential forms and exterior differentiation and Cech Cohomology allows us to extract the data of inclusions on open sets and contravariant expansion of the ring of C^∞ functions on these open sets. Again although extremely different in construction these cohomology theories turn out to be isomorphic on smooth manifolds with the standard structure sheaf $\mathcal{O}(X)$.

2 Homotopy Equivalence and Key Theorems

The goal of Algebraic Topology is to take information from topological spaces and identify algebraic structures on them in hopes to fully classify topological spaces in general.

This task first relies on a good notion of "equivalence", i.e. what is a good definition which can distinguish topological spaces in a sensible manner. To do this we turn to the notion of Homotopy Equivalence.

2.1 Homotopy Equivalence

The key aspect – really the only information we have – of a topological space is the data of its topology, i.e. its "open sets". A natural way to equivocate these collections of open sets would then be to consider functions between topological spaces which are [continuous](#) as the topological definition of continuity heavily relies on the preservation of openness. This is the foundation for the inspiration to consider topological spaces up to "continuous deformation", which will be denoted more formally as [Homotopy Equivalence](#) through the rest of the paper.

Definition 2.1. [*Homotopic maps*] Let X and Y be two topological spaces and let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be continuous maps.

f and g are said to be [homotopic](#), denoted $f \simeq g$, if there is a map $H : X \times [0, 1] \rightarrow Y$ such that:

1. $H(x, 0) = f(x)$
2. $H(x, 1) = g(x)$
3. H is continuous for all $t \in [0, 1]$ and for all $x \in X$.

Remark 2.2. Often, the map $H : X \times [0, 1] \rightarrow Y$ is denoted as a family of maps: $H_t : X \rightarrow Y$ where each H_t corresponds to some $t \in [0, 1]$.

This will be the notation used throughout the rest of the paper.

Although the definition of homotopic maps above is a bit hard to digest, with the notational changes to considering a homotopy as a family of continuous maps which changes continuously in $t \in [0, 1]$ one can get a very pictorial interpretation of the definition.

Definition 2.3 ([Homotopy Equivalence](#)). Let X and Y be topological spaces. X and Y are said to be [Homotopy Equivalent](#) if there exists continuous maps $\rho : X \rightarrow Y$ and $\psi : Y \rightarrow X$ such that:

1. $\psi \circ \rho \simeq id_X$
2. $\rho \circ \psi \simeq id_Y$

Here notice that we do not require equality on the nose between the composition of the above maps and the identity maps on the topological spaces. The notion of [homeomorphism](#) is defined the same way as above except with the altered requirement of the compositions being equal to their respective identity. This may seem to be the most natural notion of equivalence for topological spaces, however, this is too restrictive as it completely disregards a notion of "contractibility" to be

defined later. The act of degenerating the topological space to a single point is not a continuously invertible act and, thus, homeomorphisms would not capture this important property.

Remark 2.4. *It should be noted that there is a subtle distinction between homotopy equivalence and the intuitive notion of "continuous deformation".*

When one thinks of "continuous deformation" they are thinking of the manipulation of the image of a function. Although most times this is sufficient intuition, there are cases where this intuition fails and the following example showcases this failure explicitly.

Consider $S^2 \amalg D^1$:

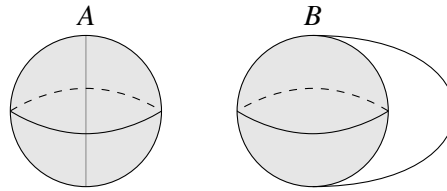


Figure 1: Both A and B are valid pictures of $S^2 \amalg D^1$, but there is no way to continuously deform A into B or vice versa because the D^1 would have to go through the S^2 .

Proposition 2.5. *Homotopy equivalence is indeed an equivalence relation on topological spaces.*

Proof. First let me use the following notation: $X \sim Y$ if X is homotopy equivalent to Y .

I will conduct this proof in the following three steps:

- (I.) $X \sim X$
- (II.) $X \sim Y \implies Y \sim X$
- (III.) $X \sim Y$ and $Y \sim Z \implies X \sim Z$

(I) Consider the identity map on X :

$$\begin{array}{ccc} Id_X: X & \longrightarrow & X \\ \Downarrow & & \Downarrow \\ x & \longmapsto & x \end{array}$$

The identity map is continuous and it suffices to show that the composition $Id_X \circ Id_X$ is homotopic to Id_X . This is trivially true, but just to make sure we don't leave any room for points to be lost consider the homotopy:

$$\begin{array}{ccc} F : X \times [0, 1] & \longrightarrow & X \\ \downarrow \Psi & & \downarrow \Psi \\ (x, t) & \longmapsto & x \end{array}$$

It follows that $F(x, t) = f_t(x) = x$ for all $t \in [0, 1]$ and $f_0 = Id_X = Id_X \circ Id_X$ and $f_1 = Id_x$. Since all maps here are vacuously continuous, it follows that F is indeed a homotopy and $X \sim X$.

(II) Let $X \sim Y$. Thus, by definition of homotopy equivalence, there exists continuous maps $f : X \rightarrow Y$, $g : Y \rightarrow X$ such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X .

To show that $Y \sim X$ we need to find continuous maps between X and Y which when composed are homotopic to their respective identity maps. Notice that we already have such maps. Take f and g from above. It follows that they define the homotopy equivalence $Y \sim X$ as well.

Lemma 1. *Let $f, g : Y \rightarrow Z$ be homotopic maps and $h : X \rightarrow Y$ be a continuous map. It follows that $f \circ h$ is homotopic to $g \circ h$. (Note the left composition statement has an identical proof)*

proof. It follows that since $f, g : Y \rightarrow Z$ are homotopic there exists a homotopy:

$$\begin{array}{ccc} F : Y \times [0, 1] & \longrightarrow & Z \\ \downarrow \Psi & & \downarrow \Psi \\ (y, t) & \longmapsto & z \end{array}$$

Such that if we define $F(y, t) = f_t(y)$, we see $f_0 = f$ and $f_1 = g$. Now consider the homotopy $H : X \times [0, 1] \rightarrow Y$ such that $H(x, t) = F(h(x), t)$. Since H is the composition of two continuous functions, H is also continuous. Also, if we define $H(x, t) = h_t(x)$ it follows that $h_0 = f \circ h$ and $h_1 = g \circ h$. Thus, we see that $f \circ h$ is homotopic to $g \circ h$. □

(III) Let $X \sim Y$ and $Y \sim Z$. It follows that there exists continuous maps $f : X \rightarrow Y$, $g : Y \rightarrow X$ and $i : Y \rightarrow Z$, $j : Z \rightarrow Y$ such that:

1. $f \circ g$ is homotopic to Id_Y
2. $g \circ f$ is homotopic to Id_X
3. $i \circ j$ is homotopic to Id_Z
4. $j \circ i$ is homotopic to Id_Y

Now define $\psi = i \circ f$ and $\rho = g \circ j$. I want to show that $\psi \circ \rho$ is homotopic to Id_Z and $\rho \circ \psi$ is homotopic to Id_X . Notice:

$$\psi \circ \rho = i \circ f \circ g \circ j$$

$$\rho \circ \psi = g \circ j \circ i \circ f$$

It then follows from the lemma and the similar fact we showed in class with left composition that:

Since $f \circ g$ is homotopic to Id_X it follows that since i is continuous, $i \circ f \circ g$ is homotopic to $i \circ Id_X$. Also, since j is continuous it follows that $i \circ g \circ g \circ j$ is homotopic to $i \circ Id_X \circ j$. Now we have that $\psi \circ \rho$ is homotopic to $i \circ Id_X \circ j = i \circ j$. Finally, since $i \circ j$ is homotopic to Id_Z we see that $\psi \circ \rho$ is homotopic to Id_Z .

Since $j \circ i$ is homotopic to Id_Y it follows that since g is continuous, $g \circ j \circ i$ is homotopic to $g \circ Id_Y$. Also, since f is continuous it follows that $g \circ j \circ i \circ f$ is homotopic to $g \circ Id_Y \circ f$. Now we have that $\rho \circ \psi$ is homotopic to $g \circ Id_Y \circ f = g \circ f$. Finally, since $g \circ f$ is homotopic to Id_X we see that $\rho \circ \psi$ is homotopic to Id_X .

Thus, by definition $X \sim Z$.

□

2.2 Cell Complexes

Now we take a bit of a detour to discuss the notion of the Cell Complex. Topological spaces in general can be a bit tough to define, however, with very basic building blocks called [cells](#) we can easily define most topological spaces that often appear in applications of topology. The nice thing about cell complexes is that they provide a lot of structure to topological spaces, allowing for manipulations to occur at the level of cells rather than global changes.

Definition 2.6 (Cell). We define a k -dimensional [cell](#), e^k , as follows:

$$e^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : \|(x_1, \dots, x_k)\| \leq 1\}$$

Remark 2.7. Notice that the definition here is identical to that of D^k . Other than to distinguish the notation there is no difference between the two objects.

We choose to embed the k -cell in the above way to easily define the "attachment maps" for cell complexes later in practice.

Definition 2.8 (Skeleton). We define the k -Skeleton to be the union of all cells of dimension less than or equal to k .

Definition 2.9 ((Finite) Cell Complex). A *cell complex* is the collection of the following data:

1. $\{E^i\}_{i=0}^n$ where for each $i \in \{0, \dots, n\}$, $E^i = \{e_\alpha^i\}_{\alpha \in A_i}$.
2. for each e_α^i we have an "attachment map", $\varphi_\alpha^i : \partial e_\alpha^i \rightarrow X^{i-1}$

As the above definition is terse we consider the following example of the cell complex of a torus. We can build the cell complex of the torus with a single zero cell, e^0 .

Thus, our zero skeleton $X^0 = e^0$.

Two one cells, e_1^1, e_2^1 , with attachment maps $\varphi_i^1 : \pm 1 \rightarrow e^0$ for $i = 1, 2$.

Thus, our one skeleton is $X^1 = X^0 \amalg_{\varphi_1^1} e_1^1 \amalg_{\varphi_2^1} e_2^1$.

Finally we need a single two cell, e^2 , with the attachment map $\varphi^2 : \partial e^2 \rightarrow X^1$ where $\partial e^2 = S^1$ is mapped to the one cells in the following order: $e_1^1 e_2^1 e_1^1 e_2^1$.

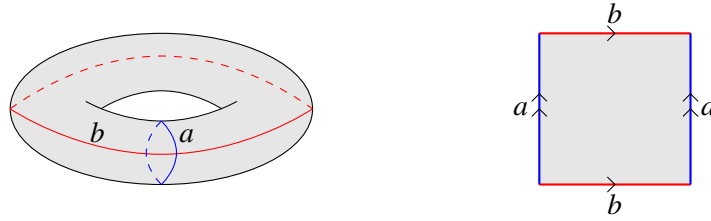


Figure 2: **Left:** T^2 with non-contractible loops **Right:** Polygonal representation of T^2

Remark 2.10. Cell decomposition is not unique. However, the cell decomposition that uses the least number of total cells is unique.

Exercise 2.11. Let G be a connected graph with V vertices and E edges. Prove that G is homotopy equivalent to a graph with 1 vertex and $E - V + 1$ edges.

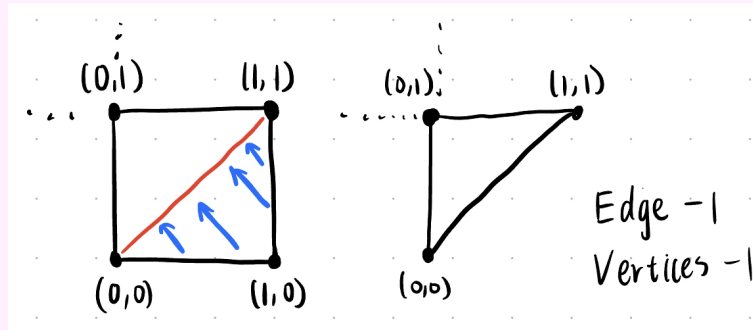


Figure 3: Exercises 2.17

Define a function $f_t : X \rightarrow Y$ such that $f_t = (x, (x - y)t + y)$. This map sends (x, y) to (x, x) , the map between the a graph of E edges and V vertices to the graph of $E - 1$ edges and $V - 1$ vertices. To show that these two graphs are homotopy equivalent, we check using the same method in 2.15:

(a) $f_t : X \rightarrow Y: f_t = (x, (x - y)t + y)$.

(b) $t \in I = [0, 1]$.

(c) $f_0 = id_X$. Plugging in $t = 0$, we get $(x, y) = (x, (x - y)0 + y) = (x, y) = id_X$.

(c) $f_1(X) = S^1$. Plugging in $t = 1$, we get $(x, y) = (x, (x - y + y)) = (x, x)$.

(d) $f_t|_Y = id_Y \forall t$. Plugging in $(x, x) = (x, (x - y)t + y)$, we get $f_t = (x, (x - x)t + x) = (x, x) = id_Y$.

We have shown that a Mobius band is homotopy equivalent to a circle.

2.3 Two Criteria for Homotopy Equivalence

Proving that two topological spaces are homotopy equivalent with its original definition is usually difficult. To help remedy this we introduce two key tools used to quickly identify homotopy equivalence: [Deformation Retraction](#) and [Quotient by a Contractible Subcomplex](#).

Deformation retraction is a tool which is often used to visually see a homotopy equivalence as a literal deformation onto a subspace, encompassing a large class of homotopy equivalences.

Quotienting by a contractible subcomplex lets us easily identify the homotopy type of a cell complex by "getting rid" of unnecessary information.

Definition 2.12. Deformation Retraction

A **deformation retraction** of a space X onto a subspace A is a homotopy $f_t : X \rightarrow X$ such that:

1. $f_0 = id_X$
2. $f_1(X) = A$
3. $f_t|_A = id_X$ for all $t \in [0, 1]$

Remark 2.13. Notice that the first two of the above conditions simply tell us that the image of the homotopy f_t is initially the entire space X and terminates to a subspace of X .

The third of the above conditions gives notion to the invariance of the subspace under the homotopy as we would like to preserve the information of the subspace being retracted to.

Proposition 2.14. Let A and B be subspaces of X , both of which are deformation retraction of X . Then $A \simeq B$.

This is a consequence of deformation retraction being a homotopy equivalence between two spaces. i.e., if a space X deformation retracts onto a subspace A , then X is homotopy equivalent to A . We also have that by Proposition 2.5, homotopy equivalence is indeed an equivalence relation, so we know $X \simeq A$ and $X \simeq B$, hence by transitivity, we know $A \simeq B$.

To illustrate this idea, we will do a quick example showing that the Mobius band is homotopy

equivalent to a circle.

Exercise 2.15. *Prove that the Mobius band is homotopy equivalent to a circle.*

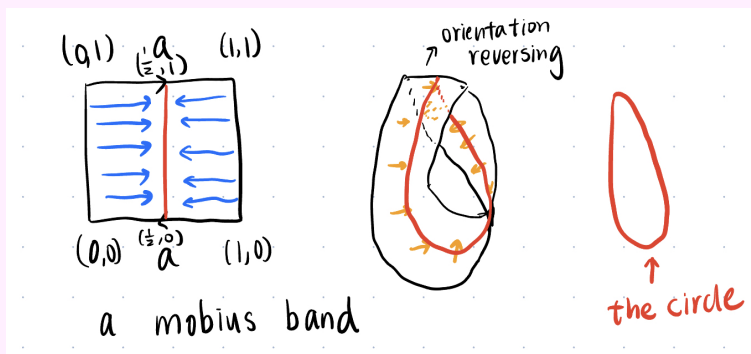


Figure 4: Exercises 2.17

Note that any two Mobius band are homeomorphic and any two circles are homeomorphic. So it is enough to show this Mobius band is homotopy equivalent to this circle to show that a Mobius band is homotopy equivalent to a circle.

Define a function $f_t : X \rightarrow Y$ such that $f_t = ((\frac{1}{2} - x)t + x, y)$. This map sends (x, y) to $(\frac{1}{2}, y)$, the map between the Mobius band and the circle.

To show it is a homotopy equivalence, we check (using the definitions of deformation retraction because homotopy equivalence is deformation retraction):

(a) $f_t : X \rightarrow Y : f_t = ((\frac{1}{2} - x)t + x, y)$

(b) $t \in I = [0, 1]$

(c) $f_0 = id_X$. Plugging in $t = 0$, we get $(x, y) = ((\frac{1}{2} - x)0 + x, y) = (x, y) = id_X$

(c) $f_1(X) = S^1$. Plugging in $t = 1$, we get $(x, y) = ((\frac{1}{2} - x)1 + x, y) = (\frac{1}{2}, y)$

(d) $f_t|_{S^1} = id_{S^1} \forall t$. Plugging in $(x, y) = (\frac{1}{2}, y)$, we get $f_t = ((\frac{1}{2} - \frac{1}{2})t + \frac{1}{2}, y) = (\frac{1}{2}, y) = id_{S^1}$.

We have shown that a Mobius band is homotopy equivalent to a circle.

One other visualization of deformation retraction is that in the figure below. If we follow along thin lines to turn the oval shaped space into a space of black thick lines, the consequence is that all of those subspaces are homotopy equivalent to each other.

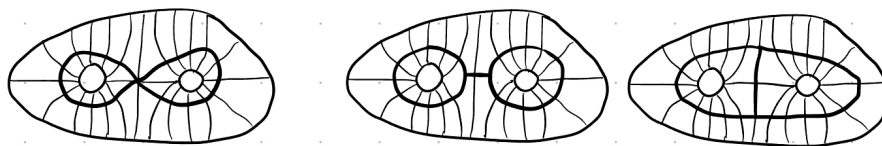


Figure 5: Deformation retractions where a disk with two smaller open subdisks removed shrinks to three different subspaces.

Remark 2.16. A **subcomplex** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X . Since A is closed, the characteristic map of each cell in A is contained in A , so A is a cell complex. A pair (X, A) consisting of a cell complex X and a subcomplex A will be

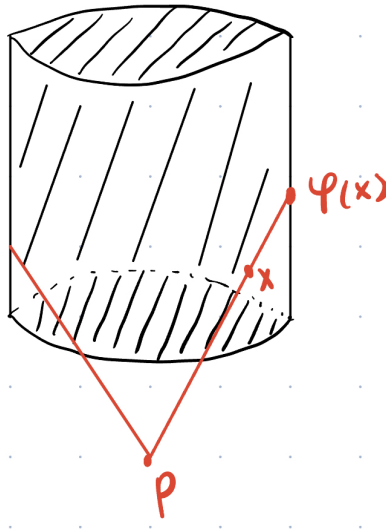
called a *CW pair*.

Theorem 2.17 (Quotient by a Contractible Subcomplex). *If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A , then the quotient map $X \rightarrow X/A$ is a homotopy equivalence.*

Before we sketch the proof of this theorem there are three key lemmas:

Lemma 2. $D^n \times [0, 1]$ deformation retracts to $D^n \times \{0\} \cup_{S^{n-1} \times \{0\}} \partial D^n \times [0, 1]$

Idea. The idea here is to take some point $p \in (D^n \times [0, 1])^C$ and construct a "straightline homotopy" to the boundary of the "cup" $D^n \times \{0\} \cup_{S^{n-1} \times \{0\}} \partial D^n \times [0, 1]$.



The homotopy which we get is $H_t(x) = x(1-t) + \rho(x)t$ where $x \in D^n \times [0, 1]$ and $\rho(x)$ is the straight line projection to $D^n \times \{0\} \cup_{S^{n-1} \times \{0\}} \partial D^n \times [0, 1]$ as above. $\square$

Lemma 3. *Let X be a finite cell complex and A be a contractible subcomplex. $X \times [0, 1]$ deformation retracts onto $X \times \{0\} \cup_A A \times [0, 1]$*

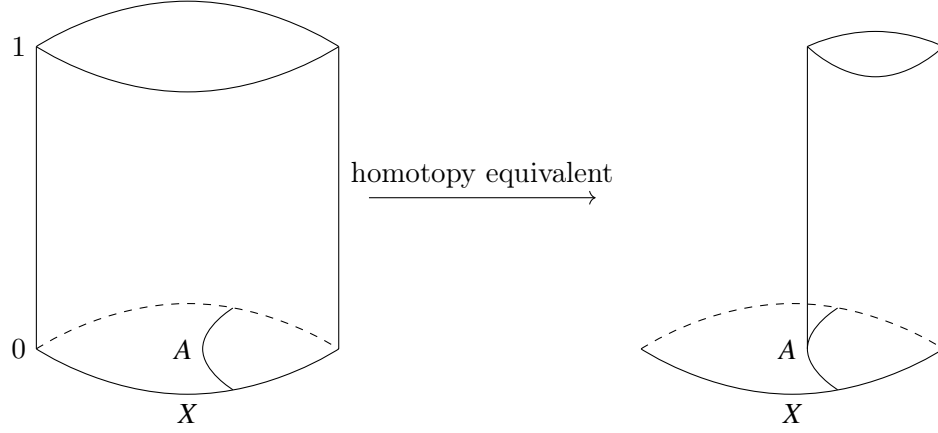


Figure 6: The proof idea is that since A is a contractible subcomplex, the two topological spaces pictured above are homotopy equivalent.

Proof. Since X is a finite CW complex, via induction on the cells of X which are not in A , using lemma 2 above we are done.

□

Lemma 4 (Homotopy Extension). *Let X be a finite cell complex and A be a contractible subcomplex. Let $f_0 : X \rightarrow Y$ be a continuous map and denote $f_0|_A := g_0$.*

Given a homotopy $g_t : A \rightarrow A$, we can extend to a homotopy $f_t : X \rightarrow Y$ such that $f_t|_A = g_t$ for all $t \in [0, 1]$.

idea. By Lemma 3 above we have that $X \times [0, 1]$ is homotopy equivalent to $X \times \{0\} \cup_A A \times [0, 1]$. Thus, since the composition of homotopies is a homotopy it suffices to provide a homotopy from $X \times \{0\} \cup_A A \times [0, 1]$ to Y .

Consider $\tilde{f}_t : X \times \{0\} \cup_A A \times [0, 1] \rightarrow Y$ such that:

$$\tilde{f}_t(x) = \begin{cases} f_0(x) & , \text{ if } x \in X \setminus A \\ g_t(x) & , \text{ if } x \in A \end{cases}$$

By construction $\tilde{f}$ is a homotopy and we are done.

□

proof of Quotient By Contractible Subcomplex. Since A is contractible, it is by definition homotopy equivalent to a point. Thus, it follows that $Id_A \simeq a_0$, where $a_0 : A \rightarrow A$ is defined as the constant map $a_0(b) = a_0$ for all $b \in A$.

By definition of homotopic maps we know there is a homotopy $g_t : A \rightarrow A$ such that $g_0 = Id_A$ and $g_1 = a_0$.

Let $f_0 = Id_X$. By Lemma 4 there is a homotopy $f_t : X \rightarrow X$ which is an extension of g_t such that $f_t|_A = g_t$ for all $t \in [0, 1]$. Namely for $t = 1$ we have $f_1|_A = g_1$.

With $q : X \rightarrow X/A$ as the natural projection map into the quotient, we find that:

1. $f_1 \circ q = f_1 \simeq f_0 = Id_X$
2. $q \circ f_1 = Id_{X/A}$

□

3 Fundamental Group

The equivalence relation of homotopy equivalence lets us classify topological spaces into equivalence classes. We want to study what things are invariant within each of these equivalence classes. Because we are studying algebraic topology, it might not be surprising to learn that the invariants we are studying are of the algebraic variety. The first one we will discuss is the Fundamental Group.

3.1 Definition

Definition 3.1. Fundamental Group

Let X be a topological space. The **Fundamental Group of X at base point x_0** is defined to be

$$\pi_1(X, x_0) := \{f : [0, 1] \rightarrow X \mid f \text{ is continuous, } f(0) = f(1) = x_0 \text{ for some } x_0 \in X\} / \simeq .$$

In words, it is the set of continuous functions from $[0, 1]$ to X that start and end at a certain point x_0 in X , up to homotopy equivalence of those functions.

The Fundamental Group, at its core, is the different ways to draw circles in a topological space. A simple example to understand this set is to look at the fundamental group of the circle itself, S^1 .

Exercise 3.2. The circle, S^1 , can be parameterized by

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$$

for $t \in [0, 1]$. This defines a closed loop on S^1 that starts at the point $(0, 1)$, so it is an element of $\pi_1(S^1, (0, 1))$. The other elements of $\pi_1(S^1, (0, 1))$ are reparameterizations of γ to make it go around many times or in the opposite direction. That is, some other valid loops are

$$\gamma_n(t) = (\cos(2\pi nt), \sin(2\pi nt)).$$

for $t \in [0, 1]$ and $n \in \mathbb{Z}$. Because these are fundamentally different ways to draw a circle, these are different elements in the fundamental group of S^1 at base point $(1, 0)$. It turns out that $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$, but a more rigorous proof depends on the use of covering spaces, which we do not have time to cover in this paper [Hat02, Thm. 1.7].

There are some important properties about π_1 , one of which is that it is indeed a group. This begs the question, “what is the binary operation?”

Definition 3.3. Product Path

Let f and g be elements in $\pi_1(X, x_0)$ for some topological space X and some $x_0 \in X$. Then the **product path** of f and g , denoted $f * g$ is the path

$$f * g := \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} .$$

Note that intuitively, the product path $f * g$ is simply the path that goes around f then goes

around g , where each is completed in half the regular “time.” Also notice that $f * g$ is an element of $\pi_1(X, x_0)$, since

$$(f * g)(0) = f(0) = g(1) = (f * g)(1) = x_0,$$

and both f and g are continuous individually. This means that it is a closed loop in the topological space that starts and ends at the base point. This is the binary operation on $\pi_1(X, x_0)$.

Proposition 3.4. *Let X be a path-connected topological space. Then $\pi_1(X, x_0)$ is a group with binary operation $*$, the product path.*

Proof. We will show that there is an identity, that the binary operation is associative, and that there are inverses.

First, the identity in $\pi_1(X, x_0)$ is the equivalence class of the constant function, $f(t) = x_0$ for all t . Let $g \in \pi_1(X, x_0)$. Then $f * g \simeq g$ with the family

$$F_s(t) = \begin{cases} x_0 & \text{if } 0 \leq t \leq \frac{1}{2}(1-s) \\ g(t) & \text{if } \frac{1}{2}(1-s) \leq t \leq 1 \end{cases}.$$

Note that $F_0(t) = (f * g)(t)$, and $F_1(t) = g(t)$, and it is obvious that F_s is continuous in s , since linear functions of s had better be continuous. Similarly, for $g * f \simeq g$, we have the family

$$G_s(t) = \begin{cases} g(t) & \text{if } 0 \leq t \leq \frac{1}{2}(s+1) \\ x_0 & \text{if } \frac{1}{2}(s+1) \leq t \leq 1 \end{cases}.$$

Again, we can check that $G_0 = g * f$ and that $G_1 = g$, and it is also obvious that G_s is continuous in s . Therefore the constant function is indeed the identity element.

Next, we check that product paths are associative. Let $f, g, h \in \pi_1(X, x_0)$. We will show

$$(f * g) * h \simeq f * (g * h).$$

First, define $p(s) = \frac{1}{4}s + \frac{1}{4}$, and $q(s) = \frac{1}{4}s + \frac{1}{2}$. Notice that

$$\begin{aligned} p(0) &= \frac{1}{4}, \\ p(1) &= \frac{1}{2}, \\ q(0) &= \frac{1}{2}, \text{ and} \\ q(1) &= \frac{3}{4}. \end{aligned}$$

This is important because p and q will define the families of functions when showing associativity. We write the family of functions relating $(f * g) * h$ to $f * (g * h)$. Consider the family

$$H_s(t) = \begin{cases} f\left(\frac{1}{p(s)}t\right) & \text{if } 0 \leq t \leq p(s) \\ g(4(t - p(s))) & \text{if } p(s) \leq t \leq q(s) \\ h\left(\frac{1}{1-q(s)}(t - q(s))\right) & \text{if } q(s) \leq t \leq 1 \end{cases}.$$

This construction is somewhat natural because you essentially scale t by the difference in endpoints and translate t so that it starts at 0 for each function f, g , and h . We see that

$$H_0(t) = \begin{cases} f(4t) & \text{if } 0 \leq t \leq \frac{1}{4} \\ g(4t - 1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ h(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

which is exactly equal to $f * (g * h)$. Similarly, $H_1(t)$ is equal to $(f * g) * h$. Finally, the family H_s is continuous in s by our construction of p and q , since they are linear functions.

The final thing to show is that the fundamental group has inverses. Let f be an element in $\pi_1(X, x_0)$. We define the inverse of this to be $\bar{f}$, where $\bar{f}(t) = f(1 - t)$. The intuitive understanding for $\bar{f}$ to be the inverse of f (that is, $f * \bar{f} = \bar{f} * f$ is homotopic to the trivial loop, or the point) is to think of f as going around in its loop, then $\bar{f}$ going along the same loop but backwards until it reaches the endpoint.

Rigorously, we have $f * \bar{f}$ is homotopic to the trivial loop with the family of functions $F_s(t)$:

$$F_s(t) = \begin{cases} f(2t(1 - s)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \bar{f}(2t(1 - s) + 2s - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

As a sanity check, we see that F_0 is indeed $f * \bar{f}$. We also see that $F_1 = x_0$, since $f(2t(1 - s))$ becomes $f(0)$, and $\bar{f}(2t(1 - s) + 2s - 1)$ becomes $\bar{f}(1)$, both of which are simply x_0 . Finally, F_s is a continuous family in s , since the terms are all linear. Therefore $f * \bar{f}$ is homotopic to the trivial loop, so $\bar{f}$ is the inverse of f (we can also show a similar family for $\bar{f} * f$, but the work is the same because $f = \bar{\bar{f}}$, the inverse of the inverse). Thus we have inverses in $\pi_1(X, x_0)$.

Because $\pi_1(X, x_0)$ has an identity, $*$ is associative, and every element $f \in \pi_1(X, x_0)$ has an inverse (namely $\bar{f}$), we see that $\pi_1(X, x_0)$ is a group. $\square$

Remark 3.5. In this proof, we use “functions” in π_1 , while the true elements of π_1 are equivalence classes of functions, where the equivalence relation is homotopy.

To be more specific, we see that for classes of functions $[f]$ and $[g]$, we define $[f] * [g]$ as $[f * g]$. This is well-defined on equivalence classes because the continuous families of functions (F_s for $f \simeq f'$ and G_s for $g \simeq g'$) given by the homotopies will give us yet another continuous family of functions ($F_s * G_s$) that show $f * g \simeq f' * g'$. The rest of a more technical proof follows accordingly.

3.2 Independence of Base Point

Now that we’ve shown that $\pi_1(X, x_0)$ is a group, it is natural to then ask if we can say anything about loops on X itself, rather than loops on X that begin at a specific base point x_0 . This brings us to another important fact: $\pi_1(X, x_0) \cong \pi_1(X, y_0)$ for any two base points $x_0, y_0 \in X$ where the loops start from.

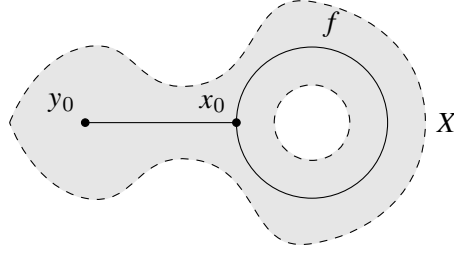


Figure 7: When showing that the fundamental group of a path-connected space does not depend on the base point, consider going along the path from y_0 to x_0 , then going around the loop at x_0 , and finally going back along the path to y_0 . This defines an element of $\pi_1(X, y_0)$ based on an element of $\pi_1(X, x_0)$.

Proposition 3.6. *Let X be a path-connected topological space. Then $\pi_1(X, x_0) \cong \pi_1(X, y_0)$ for base points $x_0, y_0 \in X$.*

Proof. Because X is path-connected, there is a path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x_0$ and $\gamma(1) = y_0$. There is also a path $\bar{\gamma} : [0, 1] \rightarrow X$ such that $\bar{\gamma}(t) = \gamma(1 - t)$. Define the map

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\longrightarrow \pi_1(X, y_0) \\ [f] &\longmapsto [\bar{\gamma} * f * \gamma]. \end{aligned}$$

First, we claim that φ is well defined. Let $[f] = [g]$. Then by definition, $f \simeq g$, so there exists the family of functions F_s such that $F_0 = f$, $F_1 = g$, and F_s is continuous with respect to s . Then $f * \gamma \simeq g * \gamma$ by the continuous mapping $F_s * \gamma$. Similarly, $\bar{\gamma} * f * \gamma \simeq \bar{\gamma} * g * \gamma$ because $\bar{\gamma} * F_s * \gamma$ is a continuous family of functions. Therefore $[\bar{\gamma} * f * \gamma] = [\bar{\gamma} * g * \gamma]$.

Next, we claim that φ is a homomorphism. Recall that $\gamma * \bar{\gamma} \simeq \bar{\gamma} * \gamma$, where both are homotopic to the “do nothing” path. Then

$$\varphi([f] * [g]) = [\bar{\gamma} * f * g * \gamma] = [\bar{\gamma} * f * \gamma * \bar{\gamma} * g * \gamma] = [\bar{\gamma} * f * \gamma] * [\bar{\gamma} * g * \gamma] = \varphi([f]) * \varphi([g]).$$

Therefore φ is a homomorphism.

Finally, we claim that φ is bijective by constructing an inverse. Define

$$\begin{aligned} \psi : \pi_1(X, y_0) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [\gamma * f * \bar{\gamma}]. \end{aligned}$$

Note that

$$\psi \circ \varphi([f]) = \psi([\bar{\gamma} * f * \gamma]) = [\gamma * \bar{\gamma} * f * \gamma * \bar{\gamma}] = [f],$$

and similarly,

$$\varphi \circ \psi([f]) = \varphi([\gamma * f * \bar{\gamma}]) = [\bar{\gamma} * \gamma * f * \bar{\gamma} * \gamma] = [f].$$

This means that $\psi = \varphi^{-1}$ by uniqueness of inverses, so φ must be bijective. A bijective homomorphism is an isomorphism, so we conclude that $\pi_1(X, x_0) \cong \pi_1(X, y_0)$. $\square$

Because of this independence of base point, we often denote the Fundamental Group of X to simply be $\pi_1(X)$, since the extra data x_0 is often unnecessary. That being said, it is important that X is path-connected for the independence of base points, so we will only consider path-connected topological spaces when we study the Fundamental Group.

One thing to note (that we will not prove) is that when X is made up of several path-connected components, say $X = \bigsqcup_{\alpha} A_{\alpha}$, where each A_{α} is path connected and disjoint from the others, we have $\pi_1(X) = \times_{\alpha} \pi_1(A_{\alpha})$.

3.3 Invariance of Homotopy Equivalence

We are now well-equipped to prove that topological spaces that are homotopy equivalent have isomorphic Fundamental groups.

Theorem 3.7. *Let X and Y be path-connected topological spaces such that X is homotopy equivalent to Y . Then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. Because X and Y are homotopy equivalent, we know there exist functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Fix $x_0 \in X$. Then define a homomorphism

$$\varphi : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0)),$$

where $\varphi(\gamma) := f \circ \gamma$. Because f and γ are continuous functions, their composition is also continuous. Because $f \circ \gamma(0) = f(x_0) = f \circ \gamma(1)$, we know that $f \circ \gamma$ is an element of $\pi_1(Y, f(x_0))$. It is obvious from its construction that φ is a homomorphism, since f takes in points from $\gamma * \gamma'$ and maps them continuously to $(f \circ \gamma) * (f \circ \gamma')$.

We claim that the inverse homomorphism is

$$\psi : \pi_1(Y, f(x_0)) \rightarrow \pi_1(X, x_0),$$

defined by $\psi(\gamma) := g \circ \gamma$. We see that

$$\psi \circ \varphi \circ \gamma_X \simeq id_X \circ \gamma_X \simeq \gamma_X,$$

so they would live in the same equivalence class of $\pi_1(X, x_0)$. Similarly, we see

$$\varphi \circ \psi \circ \gamma_Y \simeq id_Y \circ \gamma_Y \simeq \gamma_Y.$$

Therefore φ and ψ are inverses of each other, so they must be bijective homomorphisms. By independence of base point, we conclude that $\pi_1(X) \cong \pi_1(Y)$. $\square$

It makes intuitive sense that topological spaces that have similar structures will have the same kinds of loops drawn onto them, and we found the invariant we were looking for. The following sections will explore different ways to find the Fundamental group of special kinds of topological spaces. These will be less proof-heavy because an intuitive understanding is key to remembering the takeaways.

3.4 Seifert-Van Kampen Theorem

Sometimes, topological spaces X can be expressed as a union of topological spaces $\bigcup_{\alpha} A_{\alpha}$, and it is natural to wonder if one can find $\pi_1(X)$ from the various $\pi_1(A_{\alpha})$'s. It is possible, but to work our way up to defining it explicitly, we need to come up with a notion of combining two groups to make one group.

Definition 3.8. Free Product

Let $\{G_{\alpha}\}$ be a collection of groups. The **free product** of the G_{α} 's, denoted $*_{\alpha} G_{\alpha}$, is the set of all “words” $g_1 g_2 \cdots g_n$ of arbitrary length $n \geq 0$ where each g_i belongs to some G_{α} , such that adjacent letters g_i and g_{i+1} do not belong to the same group and such that g_i is not the identity element in any G_{α} .

This definition might not be particularly insightful, since “words made of group elements” is a strange concept. To clarify what this means, we will look at two important examples.

Exercise 3.9. $\mathbb{Z}/2 * \mathbb{Z}/3$

Let $\mathbb{Z}/2 = \{e_2, g\}$, where $g^2 = e$, and let $\mathbb{Z}/3 = \{e_3, h, h^2\}$. Then some words in $\mathbb{Z}/2 * \mathbb{Z}/3$ are gh^2ghg , $hghgh$, and $ghghgh \cdots ghghgh$. Notice that none of these words contain e_2 or e_3 , and they do not have adjacent letters that belong to the same group.

One thing to notice about this group is that h^2 is a single element in $\mathbb{Z}/3$, so the first word of this example still fits the criterion. We can often write the end product of composing elements of the same group into one symbol, but it is not always important to do so. That is, the “adjacency of letters” criterion in the definition is not that important.

Exercise 3.10. Let $\langle a \rangle \cong \mathbb{Z}$ and $\langle b \rangle \cong \mathbb{Z}$ be two copies of $\mathbb{Z}$ generated by a and b , respectively. Then $\langle a \rangle * \langle b \rangle$ is the set of all words made of a and b . This is not isomorphic to $\mathbb{Z}^2$ because $\mathbb{Z}^2$ would have the words made of a s and b s with the relation $ab = ba$, since $\mathbb{Z}^2$ is an abelian group.

This is an important distinction to make between the free product and the normal product of groups. The more standard product group, $\langle a \rangle \times \langle b \rangle$, is isomorphic to $\mathbb{Z}^2$ because of how elements in product groups multiply.

We care about finding $\pi_1(X)$ in relation to the $\pi_1(A_{\alpha})$ groups. Because X is path connected, we have our independence of base point in $\pi_1(X)$. This is important because doing a free product of the $\pi_1(A_{\alpha})$'s should make sense in the space, and it does. Suppose $A_{\alpha} \neq A_{\beta}$. Let $[f] \in \pi_1(A_{\alpha})$ and $[g] \in \pi_1(A_{\beta})$. Then in the free group $\pi_1(A_{\alpha}) * \pi_1(A_{\beta})$, we see $[f] * [g]$ is still well-defined because we can move the base point to $A_{\alpha} \cap A_{\beta}$. If $A_{\alpha} \cap A_{\beta}$ is empty, then we have a disjoint union, so $\pi_1(A_{\alpha} \cup A_{\beta}) = \pi_1(A_{\alpha}) \times \pi_1(A_{\beta})$, so $[f]$ and $[g]$ would not communicate via product path anyway.

When $X = \bigcup_{\alpha} A_{\alpha}$, the free product $*_{\alpha} \pi_1(A_{\alpha})$ is very close to being $\pi_1(X)$ for the reason above. The only problem is when A_{α} and A_{β} do intersect. Then loops in the intersection must be identified as one element.

Any (nonempty) intersection $A_{\alpha} \cap A_{\beta}$ gives us two natural inclusion maps: $i_{\alpha} : A_{\alpha} \cap A_{\beta} \hookrightarrow A_{\alpha}$ and $i_{\beta} : A_{\alpha} \cap A_{\beta} \hookrightarrow A_{\beta}$, where points in the intersection get mapped to the exact same points in the

space. These homomorphisms induce inclusion maps on the Fundamental groups:

$$i_{\alpha*} : \pi_1(A_\alpha \cap A_\beta) \rightarrow \pi_1(A_\alpha),$$

and

$$i_{\beta*} : \pi_1(A_\alpha \cap A_\beta) \rightarrow \pi_1(A_\beta).$$

The idea of these induced maps is that loops in the intersection can be considered as loops in each individual part. For any γ in $\pi_1(A_\alpha \cap A_\beta)$, we know that $i_{\alpha*}(\gamma)$ and $i_{\beta*}(\gamma)$ are distinct elements of $\pi_1(A_\alpha) * \pi_1(A_\beta)$. We want them to be identified as the same, so we can try quotienting the free group by the relation

$$i_{\alpha*}(\gamma)i_{\beta*}^{-1}(\gamma) = 1.$$

Note that $i_{\beta*}^{-1}(\gamma)$ is the group inverse of $i_{\beta*}(\gamma)$, not some inverse map which does not exist.

We can quotient by this relation because the set $N := \{i_{\alpha*}(\gamma)i_{\beta*}^{-1}(\gamma)\}$ forms a normal subgroup of the free group [Hat02]. This gives us the full idea of Seifert-van Kampen Theorem:

Theorem 3.11. *Let X be a path-connected topological space, and suppose $X = \bigcup_\alpha A_\alpha$ for some path-connected topological spaces A_α . Then*

$$\pi_1(X) \cong \frac{*_\alpha \pi_1(A_\alpha)}{N},$$

where N is the normal subgroup $\{i_{\alpha}(\gamma)i_{\beta*}^{-1}(\gamma)\}$.*

A proof can be found in Hatcher's Algebraic Topology, but the idea of why this works is more important [Hat02, Thm. 1.20].

3.5 Fundamental Group of a CW Complex

If a topological space X is a CW Complex, we can construct a cell decomposition $X = \bigcup e_\alpha^n$, where each e^n is an n -cell (remember that the attachment maps are very important and that the complex is constructed inductively, but it's much simpler to write $X = \bigcup e_\alpha^n$). Seifert-van Kampen Theorem gives us a simple way to compute π_1 of such unions, but there are better intuitions as to why it works.

Intuitively, we might think that the attaching maps for the 2-cells determine loops on the surface. We know that when attaching a 2-cell (denoted e^2) to the 1-skeleton of X , we need a map $\varphi : \partial e^2 \rightarrow \bigcup e_\alpha^1$, and since 2-cells are essentially 2-discs, we know that $\partial e^2 = \partial D^2 = S^1$. So it does make sense that loops in X can be determined by these boundary maps φ . That is, $\pi_1(X)$ is somehow related to the boundaries of the 2-cells.

First of all, for any boundary map φ , we see that any base point s_0 of S^1 determines a base point (namely $\varphi(s_0)$) for a loop on X . The rest of the loop in X is defined because boundary maps such as φ must be continuous. A consequence of Section 3.4 is that there is a way to compute the fundamental group of the union of topological spaces by taking an amalgamated free product of the Fundamental groups of the spaces (free product with the quotient). In this special case, the intersection of topological spaces is just the image of the boundary map the e^2 cell.

For a CW Complex, we claim that a loop in any cell is homotopic to a loop on the 1-cells. This is because loops can be continuously moved to the boundary of the cell it lives in, and those boundaries

will be lower dimensional cells. This continues until we reach the 1-dimensional boundary, which is exactly the 1-skeleton of X .

Seifert-van Kampen Theorem tells us that

$$\pi_1(X) \cong * \pi_1(e_\alpha^n) / \{i_{\alpha*}(\gamma)i_{\beta*}^{-1}(\gamma)\}.$$

As we stated above, the Fundamental groups of e^n for $n \geq 2$ will be somehow related to the Fundamental group of the 1-skeleton. In the normal subgroup $\{i_{\alpha*}(\gamma)i_{\beta*}^{-1}(\gamma)\}$, these relations will effectively remove our need to care about the higher dimensional cells. The other elements in this subgroup are the boundaries of the 2-cells that come from the attaching maps.

After removing redundant information (cells of dimension $n \geq 2$), we find that for a CW complex $X = \bigcup e_\alpha^n$,

$$\pi_1(X) = \frac{\langle e_\alpha^1 \rangle}{\{\partial e_\alpha^2\}},$$

where $\langle e_\alpha^1 \rangle$ is the free group generated by the (nontrivial) 1-cells, and $\{\partial e_\alpha^2\}$ simply identifies the boundaries of the 2-cells with their 1-cell counterparts.

Exercise 3.12. *A simple example of computing π_1 of a CW complex is the torus. We will use the polygonal representation with labels a and b for the 1-cells:*

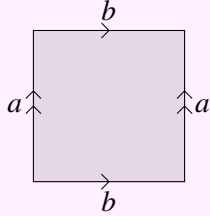


Figure 8: This is the polygonal representation of the torus, T^2 , labeled with 1-cells a and b .

We know that the fundamental group of the torus is isomorphic to $\mathbb{Z}^2$, but we want to see if we find the same thing using our new isomorphism. First, $\langle e_\alpha^1 \rangle$ is the free group generated by a and b since both are closed loops on their own (they have nontrivial π_1 's).

Next, the boundary maps for the single e^2 on the torus is $aba^{-1}b^{-1}$. (Note that this can be found by “reading the boundary” on the polygonal representation, going clockwise starting from the bottom left vertex. Any other reading of the boundary map is equivalent.) This tells us that the equivalence relation used to quotient the free group is $aba^{-1}b^{-1} = 1$, which is the same as $ab = ba$.

This gives us the abelianized free group of 2 generators, which we know is isomorphic to $\mathbb{Z}^2$, so the isomorphism works for this case as a sanity check.

4 Two Key Homology Theories

Homotopy Groups are useful for low dimensional topological spaces, but for higher dimensions, studying them using homotopy groups can be cumbersome. Here, we will introduce another topological invariant: Homology. The goal of homology theory is to create chain complex, which originated from homological algebra. Chain complex is defined as the following:

Definition 4.1. *Chain Complex*

Chain complex needs the following data:

1. $\{C_i\}_{i \in \mathbb{Z}}$, where C_i are Ring-modules
2. d_i , which is a Ring module homomorphism.

Set up of a chain complex is the following:

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

with $d_{n-1} \circ d_n = 0$

One might ask why we want a chain complex, and it is because a chain complex has information that we can use to calculate the measure of the "error" of a topological space, and we measure it by using homology. Here, an error can mean different things: one might say it is a number of n -dimensional holes, but it is more intuitive to think of it this way: Homology measures the n -dimensional difference between a given topological space and a contractible space. To use homology theory, we must define a specific chain complex structure for each homology theory. We will introduce two homology theories, simplicial and singular, and ultimately we will show that two homology groups are isomorphic.

4.1 Simplicial Homology

Simplicial homology can also be viewed as triangulating spaces, in other words patching triangles in a very specific way onto a topological space. One might ask why triangles? It is because a triangle is the "simplest" polygon we can work with. It is a convex set with the least amount of edges! As can be seen in the exercise later in this section, simplicial homology is easy to calculate with given conditions. In order to do so, we will first define what our "triangles" are, which we are going to denote as " n -simplex".

Definition 4.2. *n -simplex*

The n -simplex is the smallest convex set containing $n+1$ points. It is denoted:

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \geq 0 \text{ for all } i \right\}$$

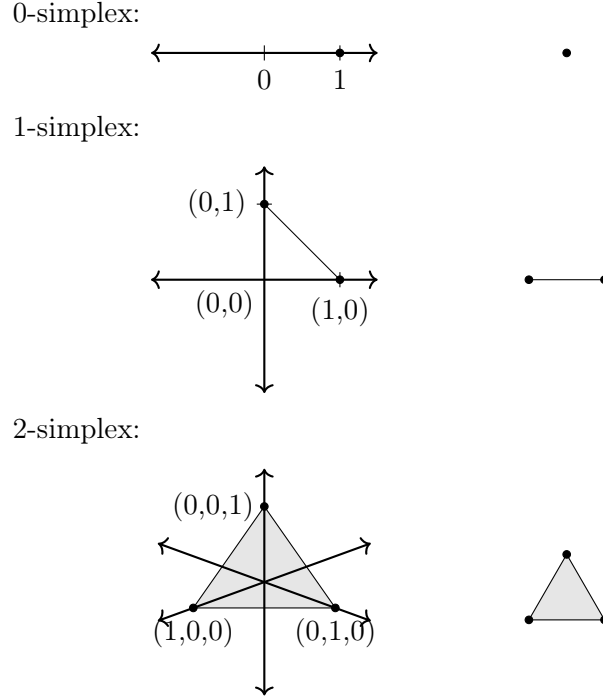


Figure 9: These are examples of n -simplices for $n = 0, 1$, and 2 .

Remark 4.3. Notice vertices of an n -simplex are:

$$\underbrace{(1, 0, \dots, 0)}_{v_0}, \underbrace{(0, 1, \dots, 0)}_{v_1}, \dots, \underbrace{(0, \dots, 0, 1)}_{v_n}$$

We may denote an n -simplex as an ordered list of vertices:

$$\Delta^n = [v_0, v_1, \dots, v_n]$$

Definition 4.4. Face

Given a simplex $\Delta^n = [v_0, v_1, \dots, v_n]$, the simplices $[v_0, \dots, \hat{v}_i, \dots, v_n]$ are called faces, where $\hat{v}_i$ denotes omitting an i^{th} vertex from a simplex.

You are removing one of the vertices from n -simplex, resulting in an $(n - 1)$ -simplex. For example, Δ^3 is a tetrahedron, and if you remove one of the vertices of a tetrahedron, you are left with a triangle, which is Δ^2 . With this example, you can also think of them as "faces of a tetrahedron is a triangle" in a geometric sense.

Remark 4.5. The union of all faces is called **boundary** of Δ^n , denoted $\partial\Delta^n$.

The **open simplex** is the space $\Delta^n - \partial\Delta^n$, denoted $\text{Int}\Delta^n$. The open simplex can be understood as "interior" of the n -simplex.

Now that we have our n -simplex defined, our next goal is to define different components of a chain complex. We will first define a rule on how to map n -simplex onto a topological space.

Definition 4.6. Δ - complex

A Δ - complex on a topological space X is a collection of maps $\{\sigma_\alpha : \Delta^n \rightarrow X\}_\alpha$ such that:

1. $\sigma_\alpha|_{Int\Delta^n}$ is injective and for each $x \in X$, x is in the image of some $\sigma_\alpha|_{Int\Delta^n}$,
2. The restriction of $\sigma_\alpha : \Delta^n \rightarrow X$ to a face agrees with some other map $\sigma_\beta : \Delta^{n-1} \rightarrow X$,
3. $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α .

Here, $Int\Delta^n$ denotes the interior of n - simplex.

1. First rule: This means when we map n -simplex onto a topological space, all of the interior of the simplex gets mapped to a topological space.
2. Second rule: This means that σ maps are telling us that we can only map "boundary to boundary". Intuitively, when we're mapping triangles, we can only map them in a way so that each edge is perfectly matched to other edges of other triangles.
3. Third rule: This rule is saying that after mapping simplices onto a topological space, that space is in quotient topology.

These restrictions on σ_α maps let us disregard cases where triangles could be on top of each other, or triangles could be patched in a way that they are touching each other only at a point. It provides the most natural way that one could think of when they are mapping triangles onto spaces.

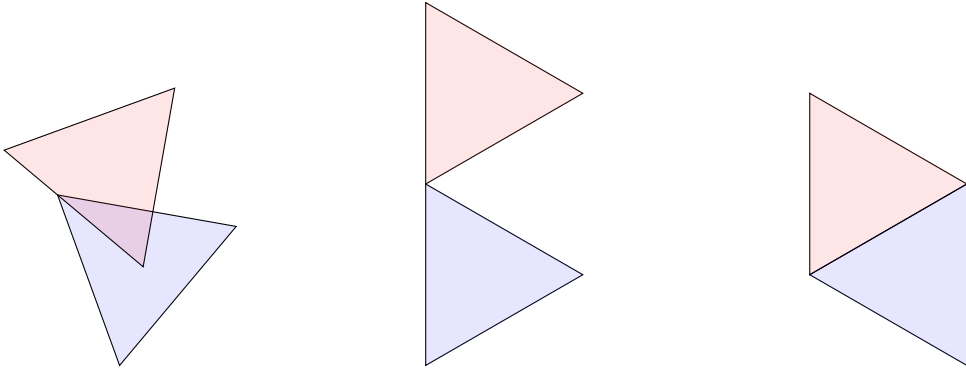


Figure 10: First two figures are bad mappings, a third figure is a good mapping

Now, we will define two data of a simplicial chain complex: an n -chain and differential maps. Differential maps tell us to read the boundaries of an n -simplex with an orientation. This may be difficult to grasp on at first, but this will become clear with an example. We will show an example after defining all the tools we need for calculating simplicial homology groups.

Definition 4.7. *n*-simplicial chain

An *n*-simplicial chain, denoted $C_n(X)$, is a free module over $\mathbb{Z}$ generated by $\{\sigma_\alpha : \Delta^n \rightarrow X\}$

Essentially, for C_n , we are getting a copy of $\mathbb{Z}$ per σ_α map.

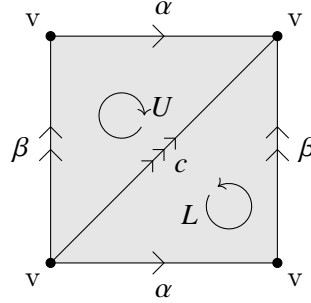


Figure 11: Polygonal representation of a torus, triangulated

For example,

$$C_1(T^2) \cong \mathbb{Z}_{\langle \sigma_\alpha \rangle} \oplus \mathbb{Z}_{\langle \sigma_\beta \rangle} \oplus \mathbb{Z}_{\langle \sigma_c \rangle}$$

Because there are three $\Delta^1 : \alpha, \beta, c$, there are three maps $\sigma_\alpha, \sigma_\beta, \sigma_c$, and we get three copies of $\mathbb{Z}$, one per each map.

Definition 4.8. *Differential (Boundary) maps*

$d_n : C_n \rightarrow C_{n-1}$ is a boundary homomorphism defined as

$$d_n(\sigma_\alpha) = \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Recall that the goal of homology is to define a chain complex, and in this case we will define simplicial chain complex. Because of our definition of boundary maps and *n*-simplicial chain, we have the data to define simplicial chain complex. One thing left to check is the condition $d_{n-1} \circ d_n = 0$.

Lemma 5. *Given boundary maps*

$$\cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

then

$$d_{n-1} \circ d_n = 0$$

Proof. We will check this condition for a generator. Given a generator $\sigma_\alpha : \Delta^n \rightarrow X$ Notice

$$\begin{aligned}
d_{n-1}(d_n(\sigma_\alpha)) &= d_{n-1} \left(\sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right) = \sum_{i=1}^n (-1)^i (d_{n-1}(\sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]})) \\
&= \sum_{i=1}^n (-1)^i \left(\sum_{j=1}^{i-1} (-1)^j \sigma_\alpha|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \underbrace{\sum_{j=i+1}^n (-1)^{j-1} \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}}_* \right) \\
&= \sum_{i=0}^n \sum_{j=0}^{i-1} (-1)^{i+j} \sigma_\alpha|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{i=0}^n \sum_{j=i+1}^n (-1)^{i+j-1} \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}.
\end{aligned}$$

Notice double sum in the left and double sum in the right differ by a factor of -1, so when added it is 0.

*: When taking the i^{th} vertex out, we are shifting $v_{i+1}, v_{i+2}, \dots, v_n$ to the left by 1, hence if $j > i$, we start the index from $i - 1$. $\square$

Now we got our simplicial chain complex. The only thing left is to define n^{th} simplicial homology group. Before defining n^{th} simplicial homology group, we must show that the image of d_n is a normal subgroup of the kernel of d_n , because n^{th} simplicial homology group is defined to be the kernel of d_n quotiented by the image of d_n . We will prove the following remark and define n^{th} simplicial homology group.

Remark 4.9. In particular, if $d_{n-1} \circ d_n = 0$, then $Im(d_n)$ is a normal subgroup of $Ker(d_{n-1})$.

Proof. First, we will prove $Im(d_n)$ is a subset $Ker(d_{n-1})$.

Assume x is an element of $Im(d_n)$. Then, we know there exists y in C_n such that $d_n(y) = x$. We want to show $d_{n-1}(x) = 0$. Notice that from the previous lemma, we know $d_{n-1} \circ d_n = 0$. So, we have the following:

$$d_{n-1}(x) = d_{n-1}(d_n(y)) = 0.$$

So x is an element of $Ker(d_{n-1})$. So, $Im(d_n)$ is a subset $Ker(d_{n-1})$.

Now we will prove $Im(d_n)$ is a subgroup of $Ker(d_{n-1})$, and to do so we will check three properties of a subgroup: Closure, Inverse, and Identity.

1. Closure: Assume x, y are elements of $Im(d_n)$. We want to show xy is an element of $Im(d_n)$. By our assumption, we know there exists x', y' such that

$$d_n(x') = x,$$

$$d_n(y') = y.$$

Since C_n is a $\mathbb{Z}$ -module, it satisfies closure property, which implies $x'y'$ is an element of C_n . Since d_n is a homomorphism, it follows that

$$d_n(x'y') = d_n(x')d_n(y') = xy.$$

So xy is an element of $Im(d_n)$.

2. Inverse: Assume x is an element of $Im(d_n)$, we want to show x^{-1} is an element of $Im(d_n)$. Since x is an element of $Im(d_n)$, there exists y in C_n such that

$$d_n(y) = x.$$

Since C_n is a $\mathbb{Z}$ -module, there exists an inverse element y^{-1} in C_n . Since d_n is a homomorphism, we have

$$d_n(y^{-1}) = (d_n(y))^{-1} = x^{-1} \in C_n.$$

Hence $Im(d_n)$ has an inverse element.

3. Identity: Since C_n, C_{n-1}, C_{n-2} are $\mathbb{Z}$ -modules, it has an identity element e, e', e'' , respectfully. Since d_n maps are group homomorphisms, we have the following:

$$d_n(e) = e',$$

$$d_{n-1}(e') = e''.$$

Which implies e' is an element of $Im(d_n)$ and an element of $Ker(d_{n-1})$. So the identity property holds.

Also notice from the identity property, it follows that both $Im(d_n)$ and $Ker(d_{n-1})$ are nonempty.

Because modules are abelian, and subgroups of abelian groups are normal, we have $Im(d_n)$ is a normal subgroup of $Ker(d_{n-1})$.

□

Definition 4.10. *n^{th} Simplicial Homology Group*

The n^{th} Simplicial Homology Group of a topological space X is a group defined as:

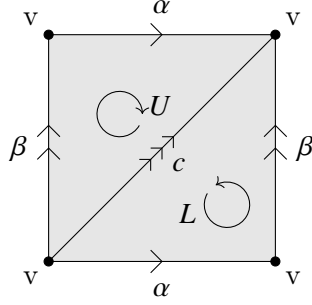
$$H_n^\Delta(X) = Ker(d_n)/Im(d_{n+1})$$

Elements of $Ker(d_n)$ are called n - cycles

Elements of $Im(d_{n+1})$ are called boundaries

One of the advantages of a simplicial homology group is that it is easy to calculate explicit homology groups of a topological space. Here, we will look at an example of calculating a simplicial homology group of a torus to illustrate how to calculate them.

Exercise 4.11. *Calculating Simplicial Homology Group of a Torus, T^2 .*



Proof. Since T^2 is made up of 0,1,and 2-simplices, we have the following chain complex:

$$0 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0.$$

By Definition 4.7, we have

$$\begin{aligned} C_0 &\cong \mathbb{Z}_{\langle \sigma_v \rangle} \\ C_1 &\cong \mathbb{Z}_{\langle \sigma_\alpha \rangle} \oplus \mathbb{Z}_{\langle \sigma_\beta \rangle} \oplus \mathbb{Z}_{\langle \sigma_c \rangle} \\ C_2 &\cong \mathbb{Z}_{\langle \sigma_U \rangle} \oplus \mathbb{Z}_{\langle \sigma_L \rangle} \end{aligned}$$

We will calculate simplicial homology groups with definition 4.1.

1. H_0^Δ : By Definition 4.1, we have

$$H_0^\Delta(T^2) = \text{Ker}(d_0) / \text{Im}(d_1).$$

Notice d_0 maps C_0 to 0, so $\text{Ker}(d_0) = C_0 \cong \mathbb{Z}_{\langle \sigma_v \rangle}$. By Definition 4.8, we have

$$\begin{aligned} \text{im}(d_1) : \\ d_1(\sigma_\alpha) &= \sigma_v - \sigma_v = 0 \\ d_1(\sigma_\beta) &= \sigma_v - \sigma_v = 0 \\ d_1(\sigma_c) &= \sigma_v - \sigma_v = 0 \end{aligned}$$

Therefore $\text{Im}(d_1) = 0$. So

$$H_0^\Delta(T^2) = \mathbb{Z}_{\langle \sigma_v \rangle} / 0 = \mathbb{Z}_{\langle \sigma_v \rangle}.$$

2. H_1^Δ : By Definition 4.1, we have

$$H_1^\Delta(T^2) = \text{Ker}(d_1) / \text{Im}(d_2).$$

From above, we have $\text{Im}(d_1) = 0$, therefore $\text{Ker}(d_1) = C_1 \cong \mathbb{Z}_{\langle \sigma_\alpha \rangle} \oplus \mathbb{Z}_{\langle \sigma_\beta \rangle} \oplus \mathbb{Z}_{\langle \sigma_c \rangle}$. For $\text{Im}(d_2)$, by Definition 4.8 we have

$$\begin{aligned} d_2(\sigma_U) &= \sigma_\beta + \sigma_\alpha - \sigma_c, \\ d_2(\sigma_L) &= \sigma_\alpha + \sigma_\beta - \sigma_c. \end{aligned}$$

Since we are quotienting by the image for simplicial homology groups, for $n, m \in \mathbb{Z}$, we can let the relation be

$$\begin{aligned}
& d_2(n\sigma_U + m\sigma_L) \\
& \stackrel{4.8}{=} nd_2(\sigma_U) + md_2(\sigma_L) \\
& \stackrel{4.8}{=} (n+m)\sigma_\alpha + (n+m)\sigma_\beta - (n+m)\sigma_c \\
& = (n+m)(\sigma_\alpha + \sigma_\beta - \sigma_c) = 0.
\end{aligned}$$

Now, we shall look at generators of $Ker(d_1)/Im(d_2)$. We have

$$\langle \sigma_\alpha, \sigma_\beta, \sigma_c \rangle / \sigma_\alpha + \sigma_\beta - \sigma_c = 0$$

Notice $\sigma_\alpha + \sigma_\beta - \sigma_c = 0 \implies \sigma_\alpha + \sigma_\beta = \sigma_c$, then above generators become $\langle \sigma_\alpha, \sigma_\beta, \sigma_\alpha + \sigma_\beta \rangle$. Since $\sigma_\alpha, \sigma_\beta$ are generators and chains are abelian, our generators become $\langle \sigma_\alpha, \sigma_\beta \rangle$. So, we have

$$H_1^\Delta(T^2) = \mathbb{Z}_{\langle \sigma_\alpha \rangle} \oplus \mathbb{Z}_{\langle \sigma_\beta \rangle}.$$

3. H_2^Δ : By Definition 4.1, we have

$$H_2^\Delta(T^2) = Ker(d_2)/Im(d_3).$$

Since the preimage of d_3 is 0, and by Definition 4.8, differential maps are homomorphism so we have $Im(d_3) = 0$. For $Ker(d_2)$, recall we have

$$\begin{aligned}
& d_2(n\sigma_U + m\sigma_L) \\
& \stackrel{4.8}{=} nd_2(\sigma_U) + md_2(\sigma_L) \\
& \stackrel{4.8}{=} (n+m)\sigma_\alpha + (n+m)\sigma_\beta - (n+m)\sigma_c \\
& = (n+m)(\sigma_\alpha + \sigma_\beta - \sigma_c) = 0.
\end{aligned}$$

Notice $(n+m)(\sigma_\alpha + \sigma_\beta - \sigma_c) = 0$ if and only if $n = -m$, so we have $Ker(d_2) = \mathbb{Z}_{\langle \sigma_\alpha + \sigma_\beta - \sigma_c \rangle}$. So we have

$$H_2^\Delta(T^2) = \mathbb{Z}_{\langle \sigma_\alpha + \sigma_\beta - \sigma_c \rangle} / 0 = \mathbb{Z}_{\langle \sigma_\alpha + \sigma_\beta - \sigma_c \rangle}.$$

We can also observe that for $n > 2$, we will have chain of 0s, hence $Ker(d_n) = 0$ which implies homology groups will be trivial. So we have

$$H_n^\Delta(T^2) = \begin{cases} \mathbb{Z}_{\langle \sigma_v \rangle}, & n = 0 \\ \mathbb{Z}_{\langle \sigma_\alpha \rangle} \oplus \mathbb{Z}_{\langle \sigma_\beta \rangle}, & n = 1 \\ \mathbb{Z}_{\langle \sigma_\alpha + \sigma_\beta - \sigma_c \rangle}, & n = 2 \\ 0, & n > 2 \end{cases}$$

□

4.2 Singular Homology

Singular homology is another kind of homology theory, where the construction is similar to that of simplicial homology, but it is more general. Recall from Definition 4.6, for simplicial homology we had restrictions on ways of mapping simplices onto a topological space. This construction is useful if our topological space is "triangulateable". As can be seen in Exercise 4.11, calculating simplicial homology group of T^2 was relatively simple because we could work with the polygonal representation (which we can triangulate). However, we are not guaranteed that our topological space will be nice enough for it to work. This is where we can use singular homology. For singular homology, we just need the mappings to be continuous. So in general, for each singular chain, we will have unaccountably many generators. It turns out singular homology becomes useful when proving theorems and lemmas, and we will see one example of that in this section.

Definition 4.12. n -Singular Chain

The n -Singular Chain is a free module over $\mathbb{Z}$ generated by $\{\sigma : \Delta^n \rightarrow X\}$, where σ is any continuous map.

Definition 4.13. n^{th} Singular Homology Group

The n^{th} Singular Homology Group of a topological space X is a group defined as:

$$H_n(X) = \text{Ker}(d_n) / \text{Im}(d_{n+1})$$

For singular homology, differential maps are defined as same as simplicial homology, and the property $d_{n-1} \circ d_n = 0$ still holds. Now, we will show an example of application of singular homology.

Proposition 4.14. Let X be a topological space, and let I be an indexing set for the number of path-connected components of X . Then $H_0(X) \cong \mathbb{Z}^I$.

Proof. Notice that $H_0(X) = \text{Ker}(d_0) / \text{Im}(d_1)$. Notice that we have the chain

$$\cdots \longrightarrow \underbrace{C_1(X)}_{=\mathbb{Z}\{\sigma : \Delta^1 \rightarrow X\}} \xrightarrow{d_1} \underbrace{C_0(X)}_{=\mathbb{Z}\{\sigma : \Delta^0 \rightarrow X\}} \xrightarrow{d_0} 0$$

Notice

$$\mathbb{Z}\{\sigma : \Delta^0 \rightarrow X\} \cong \mathbb{Z}\{x \in X\}.$$

So, we know $\text{Ker}(d_0) = \mathbb{Z}\{x \in X\}$.

We also know that

$$\text{Im}(d_1) = \mathbb{Z}\{\sigma : \Delta^1 \rightarrow X\} = \mathbb{Z}\{\sigma|_{v_1} - \sigma|_{v_0}\} = \mathbb{Z}\{x - y | x, y \text{ are path connected in } X\}.$$

Therefore, we have

$$H_0(X) = \mathbb{Z}\{x \in X | x = y \text{ if } x, y \text{ are path connected}\},$$

which is isomorphic to $\mathbb{Z}$ generated by path connected components of X . □

Remark 4.15. The above idea can be applied to Simplicial homology as well, resulting $H_0^\Delta(X) \cong \mathbb{Z}^I$.

4.3 $H_n^\Delta(X) \cong H_n(X)$

From the above 2 sections, we introduced simplicial and singular homology theories. Two theories seem they are applicable in different settings, but it turns out that two homologies are equivalent, hence they can be used interchangeably to calculate the homology of a topological space. It is not immediately obvious why this is true, and it is difficult to prove the equivalence between simplicial and singular homology groups. So, we will first observe equivalence of chain complexes and work our way down to showing that homology is an invariant, and then make our way down to showing the equivalence of simplicial and singular homology. First, we will define a map between chain complexes, known as chain maps, and show chain maps induce a well-defined map in homology.

Definition 4.16. Chain Map

Given 2 chain complexes $(C, d), (C', d')$

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & C_{n-2} & \xrightarrow{d_{n-2}} & \cdots \\ & & \downarrow \Phi_{n+1} & & \downarrow \Phi_n & & \downarrow \Phi_{n-1} & & \downarrow \Phi_{n-2} & & \\ \cdots & \xrightarrow{d'_{n+2}} & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \xrightarrow{d'_{n-1}} & C'_{n-2} & \xrightarrow{d'_{n-2}} & \cdots \end{array}$$

A Chain map $\Phi : (C, d) \rightarrow (C', d')$ is a collection of module maps $\Phi_n : C_n \rightarrow C'_n$ such that $d'_n \circ \Phi_n = \Phi_{n-1} \circ d_n$ for all n in $\mathbb{Z}$. In other words, the diagram commutes. Chain maps are homomorphisms as well.

Proposition 4.17. A chain map Φ induces a well-defined map Φ_* on homology H_n and H'_n where

$$\begin{aligned} \Phi_* : H_n &\rightarrow H'_n \\ \Phi_*([c]) &= [\Phi(c)]' \end{aligned}$$

Where $c \in H_n$, $[\cdot]$ is the equivalent class where elements that differ by $\text{Im}(d_{n+1})$ are equivalent, and $[\cdot]'$ is the equivalent class where elements that differ by $\text{Im}(d'_{n+1})$ are equivalent.

Proof. In order to show Φ_* is well-defined, we need to check two conditions:

1. $\Phi_n(c) \in \text{Ker}(d'_n)$, in other words, $\Phi_n(c)$ is in H'_n . Notice by assumption, we have $d_n(c) = 0$. Then we may check

$$d'_n(\Phi_n(c)) = \Phi_{n-1} \circ d_n(c) \text{ by the definition of a chain map,}$$

Hence we have

$$d'_n(\Phi_n(c)) = \Phi_{n-1} \circ d_n(c) = \Phi_{n-1}(0) = 0$$

So, $\Phi_n(c)$ is in $\text{Ker}(d'_n)$.

2. $\Phi_*[c + d_{n+1}(b)] = [\Phi_n(c)]'$. Notice we have

$$\begin{aligned}\Phi_*[c + d_{n+1}(b)] &= [\Phi(c + d_{n+1}(b))]' \\ &= [\Phi_n(c) + \Phi_n(d_{n+1}(b))]' \\ &= [\Phi_n(c) + d'_{n+1}(\Phi_{n+1}(b))]'.\end{aligned}$$

Notice $d'_{n+1}(\Phi_{n+1}(b))$ is in the image of d'_n , hence $d'_{n+1}(\Phi_{n+1}(b)) = 0$ by the equivalence relation of H'_n , which is $c \sim c + b$ if and only if $b \in \text{Im}(d_{n+1})$. So, we have $[\Phi_n(c) + d'_{n+1}(\Phi_{n+1}(b))]' = [\Phi_n(c)]'$.

Therefore, Φ_* is well-defined. $\square$

With above proposition, from chain map we defined a well-defined map in homology. However, this is not enough to show that homology is an invariant. So, we will implement a stronger condition, known as chain homotopy. Chain homotopy is an equivalence of chain maps, and we will witness that in order to show homology is an invariant, we only have to check equivalence up to chain homotopy. One can think of this as an analogy of checking invariant up to homotopy.

Proposition 4.18. *Let $f : X \rightarrow Y$ be a continuous map, we define a chain map:*

$$\begin{aligned}f_{\#,n} : C_n(X) &\longrightarrow C_n(Y) \\ \downarrow &\qquad \qquad \downarrow \\ \sum k_i \sigma_i &\longmapsto \sum k_i (f \circ \sigma_i)\end{aligned}$$

Proof. To prove $f_{\#,n}$ is a chain map, we must show $d_n^Y \circ f_{\#,n} = f_{\#,n-1} \circ d_n^X$. Let $\sigma : \Delta^n \rightarrow X$. Notice we have

$$\begin{aligned}d_n^Y \circ f_{\#,n}(\sigma) &= d_n^Y(f \circ \sigma) \stackrel{4.8}{=} \sum_{i=0}^n (-1)^i (f \circ \sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \\ f_{\#,n-1} \circ d_n^X(\sigma) &\stackrel{4.8}{=} f_{\#,n-1} \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right) = \sum_{i=0}^n (-1)^i f \circ (\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]})\end{aligned}$$

Notice $(f \circ \sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ is a precomposition and $f \circ (\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]})$ is a postcomposition. Since precomposition and postcomposition commute, we have $d_n^Y \circ f_{\#,n} = f_{\#,n-1} \circ d_n^X$. Therefore $f_{\#,n}$ is a chain map. $\square$

Definition 4.19. *Chain Homotopy*

Given two chain complexes (C, d) and (C', d') and two chain maps between them $\Phi_n, \Psi_n : (C_n, d) \rightarrow (C'_n, d')$, a chain homotopy between Φ and Ψ is a sequence of maps $\{h_n : C_n \rightarrow C'_{n+1}\}_{n \in \mathbb{Z}}$ such that $h_{n-1} \circ d_n + d'_{n+1} \circ h_n = \Psi_n - \Phi_n$.

Proposition 4.20. *If two chain maps are chain homotopic, then their induced maps on homology are equal.*

Proof. Given $c \in \text{Ker}(d_n)$, $[c] \in H_n$, let $\Phi_*, \Psi_* : H_n \rightarrow H'_n$ where

$$\begin{aligned}\Phi_*[c] &= [\Phi(c)]', \\ \Psi_*[c] &= [\Psi(c)]'.\end{aligned}$$

Notice we have

$$(\Psi_* - \Phi_*)([c]) = [\Psi(c) - \Phi(c)]',$$

and by the Definition 4.19, we have

$$[\Psi(c) - \Phi(c)] = [h_{n-1} \circ d_n(c) + d'_{n+1} \circ h_n(c)]'.$$

By our assumption $c \in \text{Ker}(d_n)$, we know $d_n(c) = 0$, so we get

$$[h_{n-1} \circ d_n(c) + d'_{n+1} \circ h_n(c)]' = [h_{n-1}(0) + d'_{n+1}(h_n(c))]'.$$

Since h_n is a homomorphism, it maps identity to identity, which implies $h_{n-1}(0) = 0$, so we get

$$[h_{n-1}(0) + d'_{n+1}(h_n(c))]' = [d'_{n+1}(h_n(c))]'.$$

Lastly we have that $d'_{n+1}(h_n(c)) \in \text{Im}(d'_{n+1})$ so $[d'_{n+1}(h_n(c))]' = [0]$ since we are quotienting by the equivalence of relation of H'_n .

Therefore, induced maps on homology are equal.

□

Remark 4.21. If $f_\#, g_\# : C(X) \rightarrow C(Y)$ are chain homotopic, induced maps $f_*, g_* : H_n(X) \rightarrow H_n(Y)$ are equal.

Proposition 4.22. If $f, g : X \rightarrow Y$ are homotopic maps, then the induced singular chain maps $f_\#, g_\# : C(X) \rightarrow C(Y)$ are chain homotopic.

Proof. Assume $f, g : X \rightarrow Y$ are homotopic maps, which implies there exists $F : X \times I \rightarrow Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$. To show $f_\#$ and $g_\#$ are chain homotopic, by Definition 4.19, we must show that

$$g_\# - f_\# = h_{n-1} \circ d_n^X + d_{n+1}^Y \circ h_n,$$

and we are looking for such h_n . Define h_n as following:

$$h_n(\sigma) := \sum_{i=0}^n (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

We can rearrange $g_\# - f_\# = h_{n-1} \circ d_n^X + d_{n+1}^Y \circ h_n$ as:

$$g_\# - f_\# - h_{n-1} \circ d_n^X = d_{n+1}^Y \circ h_n.$$

First, we can calculate $d_{n+1}^Y \circ h_n(\sigma)$ and get:

$$\begin{aligned} d_{n+1}^Y \circ h_n(\sigma) &= d_{n+1}^Y \left(\sum_{i=0}^n (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) \\ &= \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, v_j, \dots, w_n]}. \end{aligned}$$

By Proposition 4.18, we have

$$\begin{aligned} f_{\#}(\sigma) &= f \circ \sigma = F(\sigma, 0) = F \circ (\sigma \times id)|_{[v_0, \dots, v_n, w_n]}, \\ g_{\#}(\sigma) &= g \circ \sigma = F(\sigma, 1) = F \circ (\sigma \times id)|_{[v_0, w_0, \dots, w_n]}. \end{aligned}$$

Also notice that

$$\begin{aligned} h_{n-1} \circ d_n^X(\sigma) &\stackrel{4.8}{=} h_{n-1} \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, v_i, \dots, v_n]} \right) \\ &= \sum_{i < j} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, v_j, \dots, w_n]} \\ &\quad + \sum_{i > j} (-1)^{i-1} (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} \end{aligned}$$

Now, notice for the $d_{n+1}^Y \circ h_n(\sigma)$, when $i = j$, all of the terms in the sum cancels out except for

$$\underbrace{-F \circ (\sigma \times id)|_{[v_0, \dots, v_n, w_n]}}_{-f_{\#}}, \underbrace{F \circ (\sigma \times id)|_{[v_0, w_0, \dots, w_n]}}_{g_{\#}}.$$

Lastly, when $i \neq j$, the terms in the sum are exactly

$$\begin{aligned} & - \sum_{i < j} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, v_j, \dots, w_n]} \\ & - \sum_{i > j} (-1)^{i-1} (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} \\ & = -h_{n-1} \circ d_n^X(\sigma) \end{aligned}$$

Therefore $f_{\#}, g_{\#}$ are chain homotopic. □

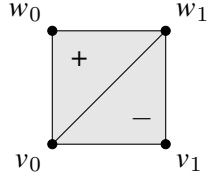
Proof for Proposition 4.22 may be overwhelming for the reader, so looking at an simple example will help the reader to understand the general idea behind the proof.

Exercise 4.23. We will consider $g_{\#} - f_{\#} = h_{n-1} \circ d_n^X + d_{n+1}^Y \circ h_n$ for $n = 1$.

Proof. We will rearrange the equation in the form $g_{\#} - f_{\#} - h_{n-1} \circ d_n^X = d_{n+1}^Y \circ h_n$. Recall since we are considering the case $n = 1$, we are dealing with 1-simplex. In this case, we have the following:

$$\begin{aligned} h_n(\sigma) &= \sum_{i=0}^1 (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \\ &= F \circ (\sigma \times id)|_{[v_0, w_0, w_1]} - F \circ (\sigma \times id)|_{[v_0, v_1, w_1]} \end{aligned}$$

Visually, we get the 2-simplex that looks like:



By Definition 4.8, we have

$$\begin{aligned}
 d_{n+1}^Y \circ h_n(\sigma) &= (F \circ (\sigma \times id)|_{[w_0, w_1]} - F \circ (\sigma \times id)|_{[v_0, w_1]} + F \circ (\sigma \times id)|_{[v_0, w_0]}) \\
 &\quad - (F \circ (\sigma \times id)|_{[v_1, w_1]} - F \circ (\sigma \times id)|_{[v_0, w_1]} + F \circ (\sigma \times id)|_{[v_0, v_1]}) \\
 &= F \circ (\sigma \times id)|_{[w_0, w_1]} - F \circ (\sigma \times id)|_{[v_0, v_1]} \\
 &\quad + F \circ (\sigma \times id)|_{[v_0, w_0]} - F \circ (\sigma \times id)|_{[v_1, w_1]}.
 \end{aligned}$$

Now notice by Proposition 4.22, we have

$$\begin{aligned}
 g_{\#}(\sigma) &= g \circ \sigma = F(\sigma, 1) = F(\sigma \times id)|_{[w_0, w_1]}, \\
 f_{\#}(\sigma) &= f \circ \sigma = F(\sigma, 0) = F(\sigma \times id)|_{[v_0, v_1]}.
 \end{aligned}$$

Notice

$$g_{\#} - f_{\#} = F \circ (\sigma \times id)|_{[w_0, w_1]} - F \circ (\sigma \times id)|_{[v_0, v_1]},$$

which is the first half of the equation. So, it is left to show that

$$-(h_{n-1} \circ d_n^X) = F \circ (\sigma \times id)|_{[v_0, w_0]} - F \circ (\sigma \times id)|_{[v_1, w_1]}.$$

Notice d_n^X takes 1-simplex as an input, so by Definition 4.8, we have

$$-(h_{n-1} \circ d_n(\sigma)) = -h_{n-1}(\sigma|_{[v_1]} - \sigma|_{[v_0]}).$$

Then we have

$$\begin{aligned}
 -h_{n-1}(\sigma|_{[v_1]} - \sigma|_{[v_0]}) &= -(F \circ (\sigma \times id)|_{[v_1, w_1]} - F \circ (\sigma \times id)|_{[v_0, w_0]}) \\
 &= F \circ (\sigma \times id)|_{[v_0, w_0]} - F \circ (\sigma \times id)|_{[v_1, w_1]},
 \end{aligned}$$

Which is what we needed. So we have $g_{\#} - f_{\#} - h_{n-1} \circ d_n^X = d_{n+1}^Y \circ h_n$.

Geometrically, $g_{\#} - f_{\#}$ represents the top and bottom edge of $d_{n+1}^Y \circ h_n$, and $-h_{n-1} \circ d_n^X$ represents two sides of $d_{n+1}^Y \circ h_n$, representing the boundary of a 2-simplex, which is what $d_{n+1}^Y \circ h_n$ is. $\square$

With Remark 4.21 and Proposition 4.22, we can obtain the following result:

Proposition 4.24. $H_n(X) \cong H_n(Y)$

If X and Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$, for all $n \in \mathbb{Z}$.

Proof. Assume X and Y are homotopy equivalent. Then there exists $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. By Proposition 4.22, we get induced singular chain maps $f_{\#,n} : C_n(X) \rightarrow C_n(Y)$ and $g_{\#,n} : C_n(Y) \rightarrow C_n(X)$. And by Remark 4.21, we get induced maps $f_{*,n} : H_n(X) \rightarrow H_n(Y)$ and $g_{*,n} : H_n(Y) \rightarrow H_n(X)$.

Without loss of generality, we will show $g_{*,n} \circ f_{*,n} = id_{H_n(X)}$ to prove $g_{*,n} \circ f_{*,n}$ is a bijection between $H_n(X)$ and $H_n(Y)$ and since $g_{*,n} \circ f_{*,n}$ is a homomorphism, $H_n(X) \cong H_n(Y)$.

Let $c \in C_n(X)$, and denote $[\cdot]$ be equivalence classes in $H_n(X)$ and $[\cdot]'$ be equivalence classes in $H_n(Y)$. By Proposition 4.17 we have

$$\begin{aligned} (g_{*,n} \circ f_{*,n})([c]) &= g_{*,n} \circ [f_{\#,n}(c)]' \\ &= [g_{\#,n}(f_{\#,n}(c))]. \end{aligned}$$

From Proposition 4.18, we get

$$\begin{aligned} [g_{\#,n}(f_{\#,n}(c))] &= [g_{\#,n}(f \circ c)] \\ &= [g \circ f \circ c] \\ &= [g \circ f]_{\#,n}(c) \\ &= (g \circ f)_{*,n}([c]). \end{aligned}$$

So, we have $g_{*,n} \circ f_{*,n}([c]) = (g \circ f)_{*,n}([c])$. Now, by homotopy equivalence, we have $g \circ f \simeq id_X$, so we have

$$(g \circ f)_{*,n} = (id_X)_{*,n}.$$

Lastly, we have

$$\begin{aligned} (id_X)_{*,n}([c]) &= [(id_X)_{\#}(c)] \\ &= [id_X \circ c] \\ &= id_{H_n(X)} \end{aligned}$$

So, we have $g_{*,n} \circ f_{*,n}([c]) = (g \circ f)_{*,n}([c]) = (id_X)_{*,n}([c]) = id_{H_n(X)}([c])$. Proof of $f_{*,n} \circ g_{*,n} = id_{H_n(Y)}$ is similar. Therefore, $f_{*,n}$ and $g_{*,n}$ are isomorphisms between $H_n(X)$ and $H_n(Y)$, hence $H_n(X) \cong H_n(Y)$. $\square$

With results from above, we defined an isomorphism between singular homology groups of two topological spaces that are homotopy equivalent. So, this shows that homology is an invariant. Now, we will show an equivalence between simplicial and singular homology, and the idea is similar: We will first look at it at a level of chain complex, except this time we will have "short exact" sequence of chain complexes. We will define the "exactness" and "shortness" of sequence, and work our way down from there.

Definition 4.25. *Exact Sequences*

Exact sequence is a collection of Ring modules M_n and module homomorphisms $f_n : M_n \rightarrow$

M_{n-1} such that

$$\cdots M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \cdots$$

where $\text{Im}(f_{n+1}) = \text{Ker}(f_n)$ for all n .

Remark 4.26. Notice that exact sequence has a very similar structure compared to that of chain complex. The difference between exact sequence and chain complex is that for exact sequence we have $\text{Im}(f_{n+1}) = \text{Ker}(f_n)$, and for chain complex we have $\text{Im}(f_{n+1}) \subset \text{Ker}(f_n)$. So intuitively, homology measures how the chain complex is different from being exact. This is analogous to where we defined homology as measurement of how a space is different from a contractible space, because if a chain complex is exact, that means that space is empty.

Definition 4.27. Short Exact Sequence

Short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where f is injective, g is surjective, and $\text{Im}(f) = \text{Ker}(g)$.

Remark 4.28. A short exact sequence of chain complexes is a sequence of short exact sequences that are compatible with differential maps,

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \xrightarrow{d_{n+2}^A} & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_n & \xrightarrow{d_n^A} & A_{n-1} \xrightarrow{d_{n-1}^A} \cdots \\ & & \downarrow i_{n+1} & & \downarrow i_n & & \downarrow i_{n-1} \\ \cdots & \xrightarrow{d_{n+2}^B} & B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} \xrightarrow{d_{n-1}^B} \cdots \\ & & \downarrow j_{n+1} & & \downarrow j_n & & \downarrow j_{n-1} \\ \cdots & \xrightarrow{d_{n+2}^C} & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \xrightarrow{d_{n-1}^C} \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where we have the following properties:

1. Commutativity:

$$\begin{aligned} i_{n-1} \circ d_n^A &= d_n^B \circ i_n \\ j_{n-1} \circ d_n^B &= d_n^C \circ j_n \end{aligned}$$

for all n .

2. Exactness: i_n is injective, j_n is surjective, $Im(i_n) = Ker(j_n)$ for all n .

Here, i and j are chain maps.

With the above remark, we get the following proposition:

Proposition 4.29. *A short exact sequence of chain complexes induces a long exact sequence in homology:*

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \rightarrow \cdots$$

Proof. From Proposition 4.17, we know i_* and j_* are well-defined maps, hence it is left to:

1. Define $\partial : H_n(C) \rightarrow H_{n-1}(A)$,
2. Show exactness of the long exact sequence.

First, we will define a well-defined map from $H_n(C)$ to $H_{n-1}(A)$. Let $[c] \in H_n(C)$, represented by $c \in Ker(d_n^C)$. Ultimately we want to show that $\partial([c]) \in H_{n-1}(A)$ and it is represented by an element of A_{n-1} in $Ker(d_{n-1}^A)$.

- Consider the diagram:

$$\begin{array}{ccccc} A_n & \xrightarrow{d_n^A} & A_{n-1} & \xrightarrow{d_{n-1}^A} & A_{n-2} \\ \downarrow i_n & & \downarrow i_{n-1} & & \downarrow i_{n-2} \\ B_n & \xrightarrow{d_n^B} & B_{n-1} & \xrightarrow{d_{n-1}^B} & B_{n-2} \\ \downarrow j_n & & \downarrow j_{n-1} & & \\ C_n & \xrightarrow{d_n^C} & C_{n-1} & & \end{array}$$

1. Notice due to surjectivity of j_n (Definition 4.27), we know there exists b in B_n such that $j_n(b) = c$.
2. Notice by commutativity (Remark 4.28), we have $d_n^C(j_n(b)) = j_{n-1}(d_n^B(b))$. By our assumption $c \in Ker(d_n^C)$ and result from 1, we have

$$d_n^C(j_n(b)) = d_n^C(c) = 0 = j_{n-1}(d_n^B(b)).$$

This implies that $d_n^B(b) \in Ker(j_{n-1})$.

3. By Definition 4.27, we know $Ker(j_{n-1}) = Im(i_{n-1})$ and by injectivity of i_{n-1} , we know there exists a unique $a \in A_{n-1}$ such that $i_{n-1}(a) = d_n^B(b)$.
4. By Lemma 5, we know

$$0 = d_{n-1}^B(d_n^B(b)),$$

and by result 3, we know

$$d_{n-1}^B(d_n^B(b)) = d_{n-1}^B(i_{n-1}(a)),$$

and by commutativity (Remark 4.28), we know

$$d_{n-1}^B(i_{n-1}(a)) = i_{n-1}(d_{n-1}(A)),$$

so we have $i_{n-1}(d_{n-1}(A)) = 0$. Since i_{n-2} is a chain map so by Definition 4.16 and Definition 4.27, it is an injective homomorphism, so

$$d_{n-1}^A(a) = 0 \implies a \in \text{Ker}(d_{n-1}^A).$$

So, we can conclude that c is represented by $a \in A_{n-1}$.

- Now, we need to show the independence of choice of b , to show that b is unique to each c . Consider the diagram:

$$\begin{array}{ccc} A_n & \xrightarrow{d_n^A} & A_{n-1} \\ \downarrow i_n & & \downarrow i_{n-1} \\ B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_n & & \\ C_n & & \end{array}$$

Suppose $j_n(b') = j_n(b) = c$ and $i_{n-1}(a') = d_n^B(b')$.

5. By Definition 4.16, j_n is a homomorphism, so we have $j_n(b') - j_n(b) = j_n(b' - b) = 0$. This implies $b' - b \in \text{Ker}(j_n)$. By Definition 4.27, we have $b' - b \in \text{Im}(i_n)$, and by injectivity, we know there exists a unique $\bar{a} \in A_n$ such that $i_n(\bar{a}) = b' - b$.
6. By commutativity (Remark 4.28), we know

$$d_n^B(i_n(\bar{a})) = d_n^B(b' - b) = i_{n-1}(d_n^A(\bar{a})).$$

7. By Definition 4.8, d_n^B is a homomorphism, so from result 6, we have

$$d_n^B(b' - b) = d_n^B(b') - d_n^B(b) \stackrel{\text{assumption}}{=} i_{n-1}(a') - i_{n-1}(a) \stackrel{\text{homomorphism}}{=} i_{n-1}(a' - a) = i_{n-1}(d_n^A(\bar{a})).$$

By injectivity (Definition 4.27), $a' - a$ is unique to $d_n^B(b') - d_n^B(b)$. This implies that $a' - a = d_n^A(\bar{a})$, so we have $[a] = [a']$. So the choice of b does not matter since it will map to the same equivalence class of a in A_{n-1} .

- Now, we will show $a' - a = 0$, to show that $\bar{a} \in \text{Ker}(d_n^A)$. Consider the diagram:

$$\begin{array}{ccccc} & & A_n & \xrightarrow{d_n^A} & A_{n-1} \\ & & \downarrow i_n & & \downarrow i_{n-1} \\ B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_{n+1} & & \downarrow j_n & & \\ C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & & \end{array}$$

Suppose $c' - c = d_{n+1}^C(\bar{c})$.

8. By surjectivity of j_{n+1} (Definition 4.27), we know there exists $\bar{b} \in B_{n+1}$ such that $j_{n+1}(\bar{b}) = \bar{c}$.
9. By assumption, we know $b' - b = d_{n+1}(\bar{b})$. Also, by surjectivity of j_n , we know there exists $c' \in C_n$ such that $j_n(b') = c'$.
10. Using result 9, we have $d_n^B(b' - b) = d_n^B(d_{n+1}^B(\bar{b})) \stackrel{5}{=} 0$. So, we have $d_n^B(b' - b) = d_n^B(b') - d_n^B(b) = 0$.
11. By injectivity of i_{n-1} (Definition 4.27), we know there exists a unique $a', a \in A_n$ such that $i_{n-1}(a') = d_n^B(b')$ and $i_{n-1}(a) = d_n^B(b)$.
12. Since i_{n-1} and d_n^B are homomorphism (Definition 4.16, Definition 4.8 respectively), we have

$$i_{n-1}(a') - i_{n-1}(a) = i_{n-1}(a' - a) = d_n^B(b') - d_n^B(b) = 0.$$

By injectivity of i_{n-1} (Definition 4.27), we can conclude $a' - a = 0$.

Therefore, $\bar{a} \in \text{Ker}(d_{n-1}^A)$. So we have a well-defined map $\partial : H_n(C) \rightarrow H_{n-1}(A)$.

Now, we will show the exactness of the long sequence we have. In order to do so, we will have to show

1. $\text{Im}(i_*) = \text{Ker}(j_*)$,
2. $\text{Im}(j_*) = \text{Ker}(\partial)$,
3. $\text{Im}(\partial) = \text{Ker}(i_*)$.

We will show them by using set inclusion. Assume we have the long sequence:

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

1. $\text{Im}(i_*) = \text{Ker}(j_*)$.

- $\text{Im}(i_*) \subset \text{Ker}(j_*)$:

Let $[a] \in H_n(A)$. By Proposition 4.17, we have $\text{Im}(i_*) = i_*([a]) = [i(a)]$.

By Definition 4.25, we have $\text{Im}(i) = \text{Ker}(j)$, so we have $j_*([i(a)]) = [j \circ i(a)]$.

Since $i(a) \in \text{Im}(i) = \text{Ker}(j)$, we have $[j \circ i(a)] = [0] = 0 \in H_n(B)$. So, we have $i_*([a]) \in \text{Ker}(j_*)$, so we have $\text{Im}(i_*) \subset \text{Ker}(j_*)$.

- $\text{Ker}(j_*) \subset \text{Im}(i_*)$: Consider the diagram:

$$\begin{array}{ccccc} & & A_n & \xrightarrow{d_n^A} & A_{n-1} \\ & & \downarrow i_n & & \downarrow i_{n-1} \\ B_{n+1} & \xrightarrow{d_{n+1}^B} & B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_{n+1} & & \downarrow j_n & & \\ C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & & \end{array}$$

Let $[b] \in \text{Ker}(j_*)$. By Proposition 4.17, we have $j_*([b]) = [j(b)] = 0 \in H_n(C)$. Since $d_{n+1}^C(c') \in H_n(C)$, we know $j_n(b) = d_{n+1}^C(c')$.

By Definition 4.27, we know j_{n+1} is surjective, hence we know there exists $b' \in B_{n+1}$ such that $j_{n+1}(b') = c'$. By commutativity and exactness, we have:

$$\begin{aligned} d_{n+1}^C(c') &= d_{n+1}^C(j_{n+1}(b')) \\ &= j_n(d_{n+1}^B(b')) \\ &= j_n(b) \implies j_n(b - d_{n+1}^B(b')) = 0 \end{aligned}$$

Therefore, we know $b - d_{n+1}^B(b') \in \text{Ker}(j_n)$.

By Definition 4.25, we have $\text{Ker}(j_n) = \text{Im}(i_n)$, and by injectivity of i_n (Definition 4.27), we know there exists a unique $a \in A_n$ such that $i_n(a) = b - d_{n+1}^B(b')$. Then, we have $[b] = [b - d_{n+1}^B(b')] = [i_n(a)]$.

Ultimately, we want $[i(a)] = i_*([a])$, so we need to check $a \in \text{Ker}(d_n^A)$.

We have $d_n^B(i_n(a)) = d_n^B(b - d_{n+1}^B(b')) = 0$. By commutativity (Remark 4.28), we have $d_n^B(i_n(a)) = i_{n-1}(d_n^A(a)) = 0$.

By injectivity of i_{n-1} (Definition 4.27), we know $d_n^A(a) = 0$, which implies $a \in \text{Ker}(d_n^A)$. So, we have $[b] \in \text{Im}(i_*)$ which implies $\text{Ker}(j_*) = \text{Im}(i_*)$.

Therefore we have $\text{Ker}(j_*) = \text{Im}(i_*)$

2. $\text{Im}(j_*) = \text{Ker}(\partial)$.

- $\text{Im}(j_*) \subset \text{Ker}(\partial)$:

Let $b \in \text{Ker}(d_n^B)$. We know $j_*([b]) \in \text{Im}(j_*)$. We know $\partial : H_n(C) \rightarrow H_{n-1}(A)$ is well-defined, so we have $\partial(j_*([b])) \in H_{n-1}(A)$ which implies $j_*([b]) \in \text{Ker}(\partial)$. So we have $\text{Im}(j_*) \subset \text{Ker}(\partial)$.

- $\text{Ker}(\partial) \subset \text{Im}(j_*)$: Consider the diagram:

$$\begin{array}{ccc} A_n & \xrightarrow{d_n^A} & A_{n-1} \\ \downarrow i_n & & \downarrow i_{n-1} \\ B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_n & & \\ C_n & & \end{array}$$

Suppose there exists $[c] \in H_n(C)$ such that $\partial([c]) = 0 \in H_{n-1}(A)$. We know from proving existence of ∂ , we know c gets represented by some element of A_{n-1} that is in $\text{Ker}(d_n^A)$.

We will denote that $\bar{a} \in A_n$, and we know $d_n^A(\bar{a}) \in H_{n-1}(A)$.

Consider $b - i_n(\bar{a})$ for some $b \in B_n$. Notice

$$\begin{aligned} d_n^B(b - i_n(\bar{a})) &= d_n^B(b) - d_n^B(i_n(\bar{a})) \\ &= d_n^B(b) - i_{n-1}(\underbrace{d_n^A(\bar{a})}_{=0}) = 0. \end{aligned}$$

Then notice by Proposition 4.17, we have $j_*([b - i_n(\bar{a})]) = [j_n(b) - j_n \circ i_n(\bar{a})]$.

By Definition 4.27, we know $\text{Ker}(j_n) = \text{Im}(i_n)$, so we have $i_n(\bar{a}) \in \text{Im}(i_n) = \text{Ker}(j_n)$, so $j_n \circ i_n(\bar{a}) = 0$. So, we have $j_*([b - i_n(\bar{a})]) = [j_n(b)] = [c]$. So we have $[c] \in \text{Im}(j_*)$, which implies $\text{Ker}(\partial) \subset \text{Im}(j_*)$.

Therefore we have $\text{Im}(j_*) = \text{Ker}(\partial)$.

3. $\text{Im}(\partial) = \text{Ker}(i_*)$.

- $\text{Im}(\partial) \subset \text{Ker}(i_*)$: Consider the diagram:

$$\begin{array}{ccc} & & A_{n-1} \\ & & \downarrow i_{n-1} \\ B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_n & & \\ C_n & & \end{array}$$

Let $[c] \in H_n(C)$. We know $\partial([c]) \in \text{Im}(\partial)$, which implies that $\partial([c]) = [a]$ for some $[a] \in H_{n-1}(A)$. By Proposition 4.17, we have

$$i_*([a]) = [i_{n-1}(a)] = [d_n^B(b)] = 0.$$

This implies $\partial([c]) \in \text{Ker}(i_*)$ and therefore $\text{Im}(\partial) \subset \text{Ker}(i_*)$.

- $\text{Ker}(i_*) \subset \text{Im}(\partial)$: Consider the diagram:

$$\begin{array}{ccc} & & A_{n-1} \\ & & \downarrow i_{n-1} \\ B_n & \xrightarrow{d_n^B} & B_{n-1} \\ \downarrow j_n & & \downarrow j_{n-1} \\ C_n & \xrightarrow{d_n^C} & C_{n-1} \end{array}$$

Let $[a] \in H_{n-1}(A)$. Suppose $[a] \in \text{Ker}(i_*)$, i.e. $i_{n-1} * ([a]) = 0$, which implies that

$$i_{n-1}(a) = d_n^B(b).$$

Notice by Remark 4.28, $d_n^C(j_n(b)) = j_{n-1}(\underbrace{d_n^B(b)}_{=i(a)}) = j_{n-1}(i_{n-1}(a))$. By Definition 4.27,

we know $j_{n-1}(i_{n-1}(a)) = 0$. So, we have $j_n(b) \in \text{Ker}(d_n^C)$, so we have $[\partial(j_n(b))] = [a]$. Hence, we have $[a] \in \text{Im}(\partial)$ which implies $\text{Ker}(i_*) \subset \text{Im}(\partial)$.

Therefore we have $\text{Im}(\partial) = \text{Ker}(i_*)$.

So, we have a long exact sequence in homology induced from short exact sequence of chain complexes. $\square$

Now that we know short exact sequence of chain complexes induces long exact sequence in homology, we will introduce an idea of relative homology groups to show that we can construct a long exact sequence of singular and simplicial homology groups.

Definition 4.30. *Relative Homology Groups*

Given a topological space X and subspace $A \subset X$, we will define relative homology of the pair $H_n(X, A)$ as the homology of a chain complex defined as follows:

$$\begin{aligned} C_n(X, A) &= C_n(X)/C_n(A) \\ d_n : C_n(A) &\rightarrow C_{n-1}(A) \end{aligned}$$

where d_n is defined the same as singular homology.

One key result we can get out of the definition of relative homology is that we can get a short exact sequence of chain complexes which looks like

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \longrightarrow 0$$

where i is an inclusion map, and j is a quotient map. By Proposition 4.29, we know we can get a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

We may construct long exact sequence of singular and simplicial homology groups like the above way, and now all we have left is to show there is an isomorphism between each simplicial and singular homology groups that match in dimensions and topological spaces. To do so, we need Excision theorem, and Five lemma.

Theorem 4.31. *Excision Theorem*

Let $Z \subset A \subset X$ with the closure of Z is a proper subset of the interior of A . Then, the inclusion

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

induces an isomorphism on homology:

$$H_n(X - Z, A - Z) \cong H_n(X, A).$$

Intuitively, the excision theorem states that the relative homology group $H_n(X, A)$ is unaffected if we remove Z from the space. Because we have the property that closure of Z is a proper subset of the interior of A , if we look at two sets $(X - Z) - (A - Z)$ and $(X - A)$, two spaces, are the same space. Hence the sigma maps that send n -simplices onto the space should match up between two homology groups. Proof can be found in Hatcher's Algebraic Topology, but the geometric intuition behind this theorem is more important [Hat02, Thm. 2.20].

Lemma 6. *The Five Lemma*

Let A_i and B_i be abelian groups such that the following diagram commutes and has exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5 \end{array}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is also an isomorphism. Here, α_i, β_i, f_i are homomorphisms.

Proof. We will prove f_3 is an isomorphism by proving it is injective and surjective.

1. f_3 is surjective:

Let $b_3 \in B_3$. We want to show that there exists $a_3 \in A_3$ such that $f_3(a_3) = b_3$. Suppose f_1, f_2, f_4, f_5 are isomorphisms. Since f_4 is surjective, there exists a_4 such that $f_4(a_4) = \beta_3(b_3)$. By commutativity and exactness, we have

$$f_5(\alpha_4(a_4)) = \beta_4(f_4(a_4)) = \beta_4(\beta_3(b_3)) = 0.$$

Since f_5 is injective, we know $\alpha_4(a_4) = 0$. By exactness, we have $a_4 \in \text{Ker}(\alpha_4) = \text{Im}(\alpha_3)$, which implies there exists $a_3 \in A_3$ such that $\alpha_3(a_3) = a_4$.

Then by commutativity, we have

$$\beta_3(b_3) = f_4(a_4) = f_4(\alpha_3(a_3)) = \beta_3(f_3(a_3)).$$

Then we have

$$\beta_3(b_3) - \beta_3(f_3(a_3)) = \beta_3(b_3 - f_3(a_3)) = 0.$$

By exactness, we have

$$b_3 - f_3(a_3) \in \text{Ker}(\beta_3) = \text{Im}(\beta_2).$$

This implies there exists some $b_2 \in B_2$ such that $\beta_2(b_2) = b_3 - f_3(a_3)$. By surjectivity of f_2 , we know there exists $a_2 \in A_2$ such that $b_2 = f_2(a_2)$. With $b_2 = f_2(a_2)$ and commutativity, we have

$$b_3 - f_3(a_3) = \beta_2(b_2) = \beta_2(f_2(a_2)) = f_3(\alpha_2(a_2)).$$

So, we have $b_3 = f_3(a_3) + f_3(\alpha_2(a_2)) = f_3(a_3 + \alpha_2(a_2))$. which implies $b_3 \in \text{Im}(f_3)$, so f_3 is surjective.

2. f_3 is injective:

Let $a_3 \in A_3$ such that $f_3(a_3) = 0$. We want to show that $a_3 = 0$ to show injectivity of f_3 . By commutativity, we know

$$\underbrace{\beta_3(f_3(a_3))}_{=0} = f_4(\alpha_3(a_3)) = 0.$$

Since f_4 is injective, we know that $\alpha_3(a_3) = 0$. Then by exactness, we know $a_3 \in \text{Ker}(\alpha_3) = \text{Im}(\alpha_2)$. So there exists $a_2 \in A_2$ such that $\alpha_2(a_2) = a_3$. By commutativity, we know that

$$f_3(\underbrace{\alpha_2(a_2)}_{=a_3}) = \beta_2(f_2(a_2)) = 0.$$

By exactness, we know $f_2(a_2) \in \text{Ker}(\beta_2) = \text{Im}(\beta_1)$. This implies there exists $b_1 \in B_1$ such that $\beta_1(b_1) = f_2(a_2)$. Since f_1 injective, we know there exists $a_1 \in A_1$ such that $f_1(a_1) = b_1$. By commutativity, we have

$$\beta_1(f_1(a_1)) = f_2(\alpha_1(a_1)) = f_2(a_2).$$

Since f_2 is an isomorphism, we know that $a_2 = \alpha_1(a_1)$. Hence we have $a_3 = \alpha_2(a_2) = \alpha_2(\alpha_1(a_1)) = 0$. Therefore, f_3 is injective.

Therefore, f_3 is an isomorphism. $\square$

Now, we have all of machinery to get to our final goal: Simplicial homology and singular homology are equivalent.

Theorem 4.32. $H_n^\Delta(X) \cong H_n(X)$

Given a topological space X , and a Δ -complex structure on X , $H_n^\Delta(X) \cong H_n(X)$ for all n .

Proof. Given a Δ -complex structure, with finite dimension, let X^k denote the k -skeleton. Notice $X^{k-1} \subset X^k$, so by Proposition 4.29 and Definition 4.30, we can construct long exact sequences of singular and simplicial homology,

$$\begin{array}{ccccccccccc} \rightarrow & H_{n+1}^\Delta(X^k, X^{k-1}) & \xrightarrow{\partial} & H_n^\Delta(X^{k-1}) & \xrightarrow{i_*} & H_n^\Delta(X^k) & \xrightarrow{j_*} & H_n^\Delta(X^k, X^{k-1}) & \xrightarrow{\partial} & H_{n-1}^\Delta(X^{k-1}) & \rightarrow \dots \\ & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 & \\ \rightarrow & H_{n+1}(X^k, X^{k-1}) & \xrightarrow{\partial} & H_n(X^{k-1}) & \xrightarrow{i_*} & H_n(X^k) & \xrightarrow{j_*} & H_n(X^k, X^{k-1}) & \xrightarrow{\partial} & H_{n-1}(X^{k-1}) & \rightarrow \dots \end{array}$$

We will show f_1, f_2, f_4, f_5 is an isomorphism and then apply five lemma to show f_3 is an isomorphism. First, we will show f_2, f_5 are isomorphisms by induction. By induction on k , assume

$$H_n^\Delta(X^{k-1}) \cong H_n(X^{k-1}).$$

Base Case: $H_0^\Delta(X^0) \cong H_0(X^0)$, because by Proposition 4.14, we know

$$\begin{aligned} H_0^\Delta(X^0) &\cong \mathbb{Z}^{|X^0|} \\ H_0(X^0) &\cong \mathbb{Z}^{|X^0|}, \end{aligned}$$

and $H_n^\Delta(X^0) \cong H_n(X^0)$, since when $n > k$, $H_n^\Delta(X^k), H_n(X^k) = 0$.

Then we only need to consider the case where $n \leq k$. By our inductive hypothesis, we know

$H_n^\Delta(X^{k-1}) \cong H_n(X^{k-1})$, so we have that f_2, f_5 are isomorphisms.

Now, we will show that f_1 and f_4 are isomorphisms. To do so, first we will observe a structure of $H_n^\Delta(X^k, X^{k-1})$. By Definition 4.30, $H_n^\Delta(X^k, X^{k-1})$ is a homology of $C_n^\Delta(X^k)/C_n^\Delta(X^{k-1})$. By Definition 4.7, this chain is a free module over $\mathbb{Z}$ generated by $\{\sigma_\alpha : \Delta^n \rightarrow X^k\}$. Because σ -maps are maps that send n -simplex onto k -skeleton, it must be the case that when $n \neq k$, the chain is 0 and when $n = k$, it is $\mathbb{Z}_{\langle \text{number of } k\text{-simplices} \rangle}$. So, our simplicial chain complex looks like the following:

$$\cdots 0 \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z}_{\langle \text{number of } k\text{-simplices} \rangle} \xrightarrow{d} 0 \xrightarrow{d} 0 \cdots$$

This implies that we have

$$H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} 0 & \text{if } n \neq k \\ \mathbb{Z}_{\langle \text{number of } k\text{-simplices} \rangle} & \text{if } n = k \end{cases}$$

Now we will look at the structure of $H_n(X^l, X^{k-1})$. First notice that X^{k-1} has a neighborhood $\mathbf{u}$ which deformation retracts onto X^{k-1} . Then (X^k, X^{k-1}) and $(X^k, \mathbf{u})$ are homotopy equivalent, so by Proposition 4.24, we know that $H_n(X^k, X^{k-1}) \cong H_n(X^k, \mathbf{u})$. Then notice that $X^{k-1} \subset \mathbf{u} \subset X^k$, then by Theorem 4.31, we know we have $H_n(X^k, \mathbf{u}) \cong H_n(X^k - X^{k-1}, \mathbf{u} - X^{k-1})$. Now notice that $X^k - X^{k-1} \cong \sqcup \text{Int}(\Delta^k)$ which is homotopy equivalent to $\sqcup D^k$ and $\mathbf{u} - X^{k-1}$ is a neighborhood of a boundary in each $\text{Int}(\Delta^k)$, hence it is homotopy equivalent to $\sqcup \partial D^k$. Hence we have $H_n(X^k, \mathbf{u}) \cong H_n(X^k - X^{k-1}, \mathbf{u} - X^{k-1}) \cong H_n(\sqcup D^k, \sqcup \partial D^k) \cong \mathbb{Z}_{\langle \text{number of } k\text{-simplices} \rangle}$. So we have

$$H_n(X^k, X^{k-1}) \cong H_n(X^k, \mathbf{u}) \cong H_n(X^k - X^{k-1}, \mathbf{u} - X^{k-1}) \cong H_n(\sqcup D^k, \sqcup \partial D^k) \cong \mathbb{Z}_{\langle \text{number of } k\text{-simplices} \rangle}.$$

Therefore we have $H_n^\Delta(X^k, X^{k-1}) \cong H_n(X^k, X^{k-1})$, hence f_1, f_4 is an isomorphism. Now, by Lemma 6, we know f_3 is an isomorphism, hence $H_n^\Delta(X) \cong H_n(X)$. This can be generalized for infinite dimensions, and the proof can be found in Hatcher's Algebraic Topology [Hat02, Thm. 2.27]. $\square$

5 Two key Cohomology Theories

5.1 A Brief Introduction to Cohomology Theory

What is Cohomology?

Cohomology is a topological invariant which you obtain via dualizing a Chain Complex. Recall that for a well-defined theory of homology, the data we needed to have was a countable collection (indexed by $\mathbb{Z}$) of R -Modules, C_i , and a collection of R -module homomorphisms, $d_i : C_i \rightarrow C_{i-1}$, satisfying $d_i \circ d_{i+1} = 0$:

$$\cdots \xrightarrow{d_{i+2}} C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

The dualization process involves the following modifications: 1) Replace C_i with $\text{Hom}(C_i, G)$, where G is some fixed abelian group and 2) Replace d_i with the dual R -module homomorphism δ^{i-1} to get the following complex:

$$\cdots \xleftarrow{\delta^{i+1}} \text{Hom}(C_{i+1}; G) \xleftarrow{\delta^i} \text{Hom}(C_i; G) \xleftarrow{\delta^{i-1}} \text{Hom}(C_{i-1}; G) \xleftarrow{\delta^{i-2}} \cdots$$

Where via contravariance of the dual operation on $\text{Hom}(\text{Hom}(C_{i-1}; G), \text{Hom}(C_i; G))$ it follows that $0 = (d_i \circ d_{i+1})^* = d_{i+1}^* \circ d_i^* = \delta^i \circ \delta^{i-1}$. We then define the cohomology groups as:

$$H^i(X; G) = \frac{\ker \delta^i}{\text{Im} \delta^{i-1}}$$

Which, as we have a chain complex with $\delta^2 = 0$, the Cohomology groups of all degrees are well-defined (proof done for Homology Groups in a previous section).

Remark 5.1. The cohomology groups $H^n(C; G)$ are solely determined by G and by the homology groups $H_n(C)$.

Although the above fact is true, it is extremely nontrivial. One would hope that you can make the natural bijection between $H^n(C; G)$ and $\text{Hom}(H_n(C); G)$ – taking inspiration from dualization – however, we are not so lucky. There is a natural homomorphism from $H^n(C; G) \rightarrow \text{Hom}(H_n(C); G)$ sending any map $\rho : C_n \rightarrow G \in H^n(C; G)$ to its restriction to $\text{Ker} d_n \subset C_n$ then projected naturally to its quotient by the normal subgroup $\text{Im} d_{n-1} \subset \text{Ker} d_n$:

$$H^n(C; G) \xrightarrow{\text{res}} \text{Hom}(\text{Ker}(d_n); G) \xrightarrow{\pi} \text{Hom}(\text{Ker}(d_n)/\text{Im}(d_{n-1}); G) = \text{Hom}(H_n(C); G)$$

$$\rho : C_n \rightarrow G \longmapsto \rho|_{\text{Ker}(d_n)} : \text{Ker}(d_n) \subset C_n \rightarrow G \longmapsto \bar{\rho} : \frac{\text{Ker}(d_n)}{\text{Im}(d_{n-1})} = H_n(C) \rightarrow G$$

Showing that this construction is the one which leads to the fact that the cohomology groups are completely determined by G and the homology groups is tursse and will not be done here. However, this statement with the **Universal Coefficient Theorem** leads to the immediate result that:

Theorem 5.2. *If a chain map Φ between chain complexes induces an isomorphism of the homology groups, then it also induces an isomorphism on cohomology groups with any coefficient G .*

Although I will not prove this theorem, it is extremely powerful in determining that the invariance of cohomology groups under changes in coefficients is given by chain maps which induce isomorphisms on homology.

This is one of the miracles of cohomology theory in that by proving its invariance under changing the coefficient G , you may use which ever coefficients suit the development of your cohomology theory in context!

For example in Floer homology, the geometric intersection number lends naturally to $\mathbb{Z}/2$ coefficients. In Hamiltonian Floer Homology, the Maslov index also lends itself naturally to $\mathbb{Z}/2$ coefficients. De Rham cohomology lends itself naturally to $\mathbb{R}$ coefficients and Cech Cohomology, Simplicial Homology, and Singular Homology lend themselves naturally to $\mathbb{Z}$ coefficients.

The Cup Product

The second miracle of Cohomology theory is the fact that there exists a well-defined multiplication of Cohomology groups of a topological space. This is a product which is not present for Homology groups. Thus, it follows that with the natural sense of direct sum as addition, Cohomology groups form a ring! This miracle is deeply rooted in the contravariance of the Cohomology functor from **Ch** to **Ab**.

Remark 5.3 (The Failure of a Product for Homology Groups). *We can actually construct a well-defined product on homology groups fairly easily, however, as we will see it will become rather trivial.*

In cooking up a product of Homology groups the natural idea one may have is to try to find maps of the following shape, taking advantage of the cross product of nicely behaved topological spaces:

$$H_i(X) \times H_j(X) \rightarrow H_{i+j}(X \times X) \rightarrow H_{i+j}(X)$$

The first half of the above product is naturally induced by the map which sends a tuple of cells in the decomposition of X , $([e_k], [e_\ell]) \in H_i(X) \times H_j(X)$ and sending it to $[e_k \times e_\ell] \in H_{i+j}(X \times X)$.

The second half of the product has a very natural definition as the map induced by any continuous projection $X \times X \rightarrow X$, which by covariance of the homology functor induces a map $H_{i+j}(X \times X) \rightarrow H_{i+j}(X)$.

The triviality comes with the loss of information from projecting $X \times X \rightarrow X$, making our constructed product extremely boring.

With the contravariance of cohomology we find that the second half of the product above is instead induced by a map $X \rightarrow X \times X$, which one can naturally imagine is defined by $d \rightarrow (d, d)$ – dubbed the diagonal map by Hatcher – and constructs for us a highly nontrivial product called the [Cup Product](#).

Definition 5.4. Take X a topological space, and R to be a commutative ring. The *cup product* is defined as follows:

$$\begin{array}{ccc} \smile : C^k(X; R) \times C^\ell(X; R) & \longrightarrow & C^{k+\ell}(X; R) \\ \smile & & \smile \\ (\varphi, \psi) & \longmapsto & \varphi \smile \psi \end{array}$$

where

$$\begin{array}{ccc} \varphi \smile \psi : & C_{k+\ell} & \longrightarrow R \\ \smile & & \smile \\ \sigma : \Delta^{k+\ell} \rightarrow X & \longmapsto & \varphi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_k, \dots, v_{k+\ell}]}) \end{array}$$

Lemma 7.

$$d(\varphi \smile \psi) = d\varphi \smile \psi + (-1)^k \varphi \smile d\psi$$

where $\varphi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$.

The above lemma is a statement of the anti-derivation properties of d , the coboundary map, over the cup product. Giving us insight to the exterior algebra of multicovectors seen in an introductory differential geometry course.

This property immediately gives us that the coboundary map and cup product are compatible in the sense that they preserve the structure of the kernel and image of d , allowing us to conclude that there is a well-defined induced product on cohomology groups given by the taking cup product followed by taking the appropriate equivalence class in cohomology:

$$H^k(X; R) \times H^\ell(X; R) \rightarrow H^{k+\ell}(X; R)$$

Corollary 5.5. As a result of R being a commutative ring and the lemma above, we have that the cup product is antisymmetric:

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$$

where $\alpha \in H^k(X; R)$, $\beta \in H^\ell(X; R)$.

5.2 Differential Topology / Riemannian Geometry Background

In hopes of constructing the De Rham cochain complex we need to take a bit of a differential geometry / topology interlude. Here I will define the basic notions of the key players in the cochain complex.

Tangent Vectors in $\mathbb{R}^n$ as Derivations

Almost all derivations of tangent planes to a surface we've seen so far take advantage of the fact that most surfaces we've seen are embedded in a Euclidean Space, thus, the goal of this chapter is to generalize the idea of the tangent plane.

Definition 5.6. [Tangent Space]

We visualize the *tangent space* $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ as the vector space of all arrows emanating from p .

The elements of $T_p(\mathbb{R}^n)$ are called *tangent vectors* (or just vectors) at p in $\mathbb{R}^n$.

Definition 5.7. [Directional Derivative]

If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector at p , the *directional derivative* of f in the direction v at p is defined to be

$$D_v f = \lim_{t \rightarrow 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t))$$

From the chain rule we see:

$$D_v f = \sum_i \frac{df}{dx^i}(c^i(t)) \cdot \left. \frac{dc^i}{dt}(t) \right|_{t=0} = \sum_i \frac{df}{dx^i}(p^i) \cdot \left. \frac{dc^i}{dt}((0)) \right|_{t=0} = \sum_i v^i \frac{\partial f}{\partial x^i}(p)$$

Thus, we write the map that sends a function f to the number $D_v f$ by:

$$D_v = \sum_i v^i \left. \frac{\partial}{\partial x^i} \right|_p$$

usually we omit the p as it is clear that it is evaluated there.

Germes of Functions

Remark 5.8. As long as two functions agree on some neighborhood of p , then they will have the same directional derivatives at p .

This hints at the idea that we should have some sort of equivalence relation on C^∞ functions defined in some neighborhood of p .

Definition 5.9. [Germs]

Consider the set of all pairs (f, U) where U is a neighborhood of p and $f : U \rightarrow \mathbb{R}$ is a C^∞ function.

We say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W . (Shown in figure 1)

The equivalence class of (f, U) is called the *germ* of f at p . We write the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p as $C_p^\infty(\mathbb{R}^n)$.

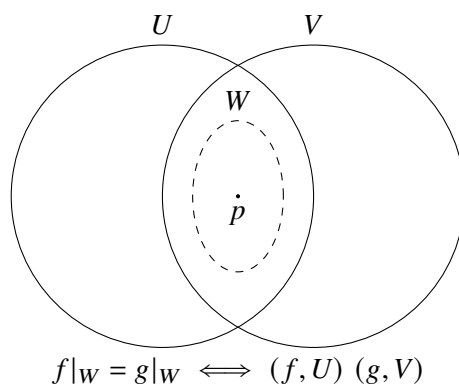


Figure 12

Derivations at a Point

Remark 5.10. For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces

$$D_v : C_p^\infty \rightarrow \mathbb{R}$$

which satisfies the *Leibniz rule* (you can see this from the chain rule definition earlier):

$$D_v(fg) = (D_v f)g(p) + f(p)D_v g$$

Definition 5.11. *[Derivation at p / Point-Derivation of C_p^∞]*

Any linear map $D : C_p^\infty \rightarrow \mathbb{R}$ satisfying the Leibniz rule is called a *derivation at p or a point-derivation of C_p^∞* .

We denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space.

Remark 5.12. Since directional derivatives at p are all derivations at p we have a natural map:

$$\begin{array}{ccc} \phi : T_p(\mathbb{R}^n) & \xrightarrow{\quad \quad} & \mathcal{D}_p(\mathbb{R}^n) \\ \downarrow \Psi & & \downarrow \Psi \\ v & \longmapsto & D_v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p \end{array} \quad (1)$$

Since D_v is linear in v , ϕ is a linear map of vector spaces.

Theorem 5.13. The linear map $\phi : T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ defined in (1) is an isomorphism of vector spaces.

proof. Note that ϕ is a linear map with respect to the addition operation on the respective spaces. Thus, it is clearly a vector space homomorphism. Thus, it remains to show that it is a bijection.

Note that ϕ is injective if the kernel is identically $\{0\}$. To prove injectivity we first suppose $D_v = 0$ for $v \in T_p(\mathbb{R}^n)$ (Thus, $v \in \text{Ker}(\phi)$). It then follows that applying D_v to any coordinate function yields:

$$0 = D_v(x^j) = \sum_i v^i \frac{\partial}{\partial x^i} \Big|_p x^j = \sum_i v^i \delta_{ij} = v^j$$

Thus, if $D_v = 0$, then v can only be 0 and $\text{Ker}(\phi) = \{0\}$.

To prove surjectivity we assume that D is some derivation at p . I want to show that there is some v such that $D = D_v$.

Let (f, V) be a representative of a germ in C_p^∞ . Since V is a neighborhood, there is some open ball $\tilde{V} \subset V$. By Taylor's Theorem with remainder there are functions $g_i \in C^\infty$ in a neighborhood of p such that

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x), \text{ where } g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

Now we apply D on both sides of the above equation. Note that $D(f(p)) = 0$ and $D(p^i) = 0$ by previous lemma.

$$Df(x) = D\left(\sum (x^i - p^i)g_i(x)\right)$$

Thus, by Leibniz rule:

$$\begin{aligned} Df(x) &= \sum (Dx^i - Dp^i)g_i(x) \Big|_{x=p} + \sum (x^i - p^i) \Big|_{x=p} Dg_i(x) \\ \implies Df(x) &= \sum (Dx^i)g_i(p) + \sum (p^i - p^i)Dg_i(x) \\ \implies Df(x) &= \sum (Dx^i) \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

Thus, we see that if $v = \langle Dx^1, \dots, Dx^n \rangle$, then $\phi(v) = D_v = D!$

□

Remark 5.14. *This theorem shows that we can bijectively identify tangent vectors and derivations at a point p .*

Under the isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $e_1, \dots, e_n$ for $T_p(\mathbb{R}^n)$ corresponds to the set $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p$.

From now on we will make this identification and write a tangent vector $v = \langle v^1, \dots, v^n \rangle = \sum v^i e_i$ as

$$v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p$$

Tangent vectors at p are derivations at p !

Vector Fields

Definition 5.15. *[Vector Field]*

A **vector field** X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$.

Since $T_p(\mathbb{R}^n)$ has a basis $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}$, the vector X_p is a linear combination

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i} \Big|_p, \quad p \in U, \quad a^i(p) \in \mathbb{R}$$

Omitting p , we may write $X = \sum a^i \frac{\partial}{\partial x^i}$, where the a^i are now functions on U .

We say that the vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on U .

Remark 5.16. One can identify vector fields on U with column vectors of C^∞ functions on U :

$$X = \sum a^i \frac{\partial}{\partial x^i} \iff \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

Remark 5.17. The ring of C^∞ functions on an open set U is commonly denoted by $C^\infty(U)$ or $\mathcal{F}(U)$.

Multiplication of vector fields by functions on U is defined pointwise:

$$(fX)_p = f(p)X_p, \quad p \in U$$

You can think of f as "scalars" and X as "vectors". Clearly, if $f \in C^\infty(U)$ then $fX = \sum (fa^i) \frac{\partial}{\partial x^i}$ is a C^∞ vector field on U .

The set of all C^∞ vector fields on U , denoted $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(U)$.

Vector Fields as Derivations

Remark 5.18. If X is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U , we define a new function Xf on U by:

$$(Xf)(p) = X_p f \quad \forall p \in U$$

Writing $X = \sum a^i \frac{\partial}{\partial x^i}$, we get

$$(Xf)(p) = \sum a^i(p) \frac{\partial f}{\partial x^i}(p)$$

or

$$Xf = \sum a^i \frac{\partial f}{\partial x^i}$$

which shows that Xf is a C^∞ function on U . Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$ -linear map

$$\begin{array}{ccc} : C^\infty(U) & \longrightarrow & C^\infty(U) \\ \downarrow \Psi & & \downarrow \Psi \\ f & \longmapsto & Xf \end{array}$$

Exercise 5.19. *[Vector Fields]*

Let X be the vector field $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf .

proof. By definition Xf is given by:

$$\begin{aligned} Xf &= \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) (x^2 + y^2 + z^2) \\ &= x \frac{df}{dx} + y \frac{df}{dy} + 0 \frac{df}{dz} \\ &= 2x^2 + 2y^2 \end{aligned}$$

□

Proposition 5.20. *[Leibniz Rule for a Vector Field]*

If X is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $X(fg)$ satisfies the product rule (Leibniz rule):

$$X(fg) = (Xf)g + f(Xg)$$

proof. At each point $p \in U$, the vector X_p satisfies the Leibniz rule:

$$X_p(fg) = (X_p f)g(p) + f(p)X_p g$$

As p varies over U , this becomes an equality of functions

$$X(fg) = (Xf)g + fXg$$

This really follows by the linearity of the partial derivative operation.

□

Remark 5.21. If A is an algebra over a field K , a derivation of A is a K -linear map $D : A \rightarrow A$ such that

$$D(ab) = (Da)b + aDb \quad \forall a, b \in A$$

Remark 5.22. The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted $Der(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map:

$$\begin{array}{ccc} \varphi: \mathfrak{X}(U) & \longrightarrow & Der(C^\infty(U)) \\ \downarrow \Psi & & \downarrow \Psi \\ X & \longmapsto & (f \mapsto Xf) \end{array}$$

5.3 Differential Forms on $\mathbb{R}^n$

Differential 1-forms and the Differential of a Function

Definition 5.23. [Cotangent Space]

The *cotangent space* to $\mathbb{R}^n$ at p , denoted $T_p^*(\mathbb{R}^n)$, is defined to be the dual space $(T_p\mathbb{R}^n)^\vee$.

Thus, an element of the cotangent space T_p^* is a covector or linear functional on the tangent space.

Definition 5.24. [Covector Field / Differential 1-form]

In parallel with a vector field, a *covector field* or *differential 1-form* on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point $p \in U$ a covector $\omega_p \in T_p^*(\mathbb{R}^n)$

$$\begin{array}{ccc} \omega: U & \longrightarrow & \bigcup_{p \in U} T_p^*(\mathbb{R}^n) \\ \downarrow \Psi & & \downarrow \Psi \\ p & \longmapsto & \omega_p \in T_p^*(\mathbb{R}^n) \end{array}$$

Remark 5.25. In the union $\bigcup_{p \in U} T_p^*(\mathbb{R}^n)$ the sets $T_p^*(\mathbb{R}^n)$ are all disjoint.

Definition 5.26. [Differential of a Function]

From any C^∞ function $f: U \rightarrow \mathbb{R}$, we can construct a 1-form df , called the *differential of f* as follows. For $p \in U$ and $X_p \in T_p U$, define

$$(df)_p(X_p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p)$$

Remark 5.27. [Unpack]

The differential form df , where $f \in C^\infty(U)$, is given explicitly by:

$$\begin{array}{ccc} df: U & \longrightarrow & \bigcup_{p \in U} T_p^*(\mathbb{R}^n) \\ \downarrow \wr & & \downarrow \wr \\ p & \longmapsto & (df)_p \in T_p^*(\mathbb{R}^n) \end{array}$$

where

$$\begin{array}{ccc} (df)_p: & T_p(\mathbb{R}^n) & \longrightarrow \mathbb{R} \\ \downarrow \wr & & \downarrow \wr \\ X_p = \sum a^i(p) \frac{\partial}{\partial x^i} \Big|_p & \longmapsto & X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}(p) \end{array}$$

Remark 5.28. The directional derivative of a function in the direction of a tangent vector at p sets up a bilinear pairing

$$\begin{array}{ccc} : T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) & \longrightarrow & \mathbb{R} \\ \downarrow \wr & & \downarrow \wr \\ (X_p, f) & \longmapsto & \langle X_p, f \rangle = X_p f \end{array}$$

You can think of a tangent vector as a function of the second argument of this pairing $\langle X_p, \cdot \rangle$.

And think of the differential $(df)_p$ at p is a function on the first argument of the pairing:

$$(df)_p = \langle \cdot, f \rangle$$

Remark 5.29. Recall that if we let $x^1, \dots, x^n$ be the standard coordinates on $\mathbb{R}^n$, then the set $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p\}$ is a basis for the tangent space $T_p(\mathbb{R}^n)$.

Proposition 5.30. If $x^1, \dots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n$, $\{(dx^1)_p, \dots, (dx^n)_p\}$ is the basis for the cotangent space $T_p^*(\mathbb{R}^n)$ dual to the basis $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p\}$ for the tangent space $T_p(\mathbb{R}^n)$.

proof. By definition

$$(dx^i)_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial}{\partial x^j} \Big|_p x^i = \delta_{ij}$$

□

Remark 5.31. The previous proposition tells us that if ω is a 1-form on an open subset U of $\mathbb{R}^n$, then at each $p \in U$, ω can be written as:

$$\omega_p = \sum a_i(p)(dx^i)_p$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U , the coefficients a_i becomes functions on U and we write $\omega = \sum a_i dx^i$.

Remark 5.32. The covector field ω is said to be C^∞ on U if coefficient functions a_i are all C^∞ on U .

Proposition 5.33. *[The differential in terms of coordinates]*

If $f : U \rightarrow \mathbb{R}$ is C^∞ on an open set U in $\mathbb{R}^n$, then

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

Proof. By previous proposition, at each $p \in U$:

$$(df)_p = \sum a_i(p)(dx^i)_p$$

Thus, $df = \sum a_i dx^i$ for some real functions a_i on U . It then follows that a_j is given by:

$$\begin{aligned} df \left(\frac{\partial}{\partial x^j} \right) &= \sum_i a_i dx^i \left(\frac{\partial}{\partial x^j} \right) \\ &= \sum_i a_i \left(\frac{\partial x^i}{\partial x^j} \right) \\ &= \sum_i a_i \delta_{ij} = a_j \end{aligned}$$

Since $df \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial f}{\partial x^j}$ it follows that

$$a_j = \frac{\partial f}{\partial x^j}$$

□

Differential k -forms

Definition 5.34. *[Differential k -form]*

A differential form ω of degree k or a k -form on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U an alternating k -linear function on the tangent space $T_p(\mathbb{R}^n)$ i.e. $\omega_p \in A_k(T_p\mathbb{R}^n)$.

Definition 5.35. We denote the vector space of C^∞ k -forms on U as $\Omega^k(U)$.

A 0-form on U assigns to each point p in U an element of $A_0(T_p\mathbb{R}^n) = \mathbb{R}$. Thus, a 0-form is just a C^∞ function on U .

Remark 5.36. The wedge product of a k -form ω and an ℓ -form τ on an open set U is defined pointwise

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p, \quad p \in U$$

In terms of coordinates, if $\omega = \sum_I a_I dx^I$ and $\tau = \sum_J b_J dx^J$, then

$$\omega \wedge \tau = \sum_{I,J} (a_I b_J) dx^I \wedge dx^J$$

Notice that if I and J are not disjoint, then $dx^I \wedge dx^J = 0$.

This shows that the wedge product of two C^∞ forms is C^∞ . So the wedge product is a bilinear map

$$\wedge : \Omega^k(U) \times \Omega^\ell(U) \rightarrow \Omega^{k+\ell}(U)$$

Differential Forms as Multilinear Functions on Vector Fields

Definition 5.37. If ω is a C^∞ 1-form and X is a C^∞ vector field on an open set U in $\mathbb{R}^n$, we define a function $\omega(X)$ on U by the formula:

$$\omega(X)_p = \omega_p(X_p), \quad p \in U$$

Remark 5.38. In coordinates

$$\omega = \sum a_i dx^i, \quad X = \sum b^j \frac{\partial}{\partial x^j} \text{ for some } a_i, b^j \in C^\infty(U)$$

so

$$\omega(X) = \left(\sum a_i dx^i \right) \left(\sum b^j \frac{\partial}{\partial x^j} \right) = \sum a_i b^i$$

Thus, $\omega(X)$ is C^∞ on U .

Remark 5.39. The previous remark tells us a C^∞ 1-form on U gives rise to a linear map from $\mathfrak{X}(U)$ to $C^\infty(U)$ over the ring $C^\infty(U)$. Thus, if $f \in C^\infty(U)$, then $\omega(fX) = f\omega(X)$. To show this, it suffices to evaluate $\omega(fX)$ at an arbitrary point $p \in U$:

$$\begin{aligned} (\omega(fX))_p &= \omega_p(f(p)X_p) \text{ (definition of } \omega(fX)) \\ &= f(p)\omega_p(X_p) \text{ (} \omega_p \text{ is } \mathbb{R}\text{-linear)} \\ &= (f\omega(X))_p \text{ (definition of } \omega(fX)) \end{aligned}$$

Remark 5.40. If we let $\mathfrak{X}(U) = C^\infty(U)$, a 1-form ω on U gives rise to an $\mathcal{F}(U)$ -linear map $\mathfrak{X}(U) \rightarrow \mathcal{F}(U)$, $X \mapsto \omega(X)$. Similarly a k -form ω on U gives rise to a k -linear map over $\mathcal{F}(U)$,

$$\begin{array}{ccc} \mathfrak{X}(U) \times \dots \times \mathfrak{X}(U) & \xrightarrow{\quad} & \mathcal{F} \\ \downarrow \Psi & & \downarrow \Psi \\ X_1, \dots, X_k & \longmapsto & \omega(X_1, \dots, X_k) \end{array}$$

The Exterior Derivative

Definition 5.41. [Exterior Derivative of 0-forms]

The exterior derivative of a C^∞ function $f \in C^\infty(U)$ is defined to be its differential $df \in \Omega^1(U)$; in terms of coordinates we see

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

Definition 5.42. [Exterior Derivative of k -forms]

For $k \geq 1$, if $\omega = \sum_I a_I dx^I \in \Omega^k(U)$, then

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \in \Omega^{k+1}(U)$$

Exercise 5.43. Let ω be the 1-form $fdx + gdy$ on $\mathbb{R}^2$, where $f, g \in C^\infty(\mathbb{R}^2)$. It follows that:

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= (g_x - f_y) dx \wedge dy \end{aligned}$$

Exercise 5.44. *[A 1-form on $\mathbb{R}^3$]*

Let ω be the 1-form $zdx - dz$ and let X be the vector field $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

solution. To calculate $\omega(X)$ we simply sum the coefficient functions of the basis of the cotangent space and tangent space as follows:

$$\omega(X) = z \cdot ydx \frac{\partial}{\partial x} + 0 \cdot xdy \frac{\partial}{\partial y} - 1 \cdot 0dz \frac{\partial}{\partial z} = zy \frac{\partial x}{\partial x} = zy$$

To calculate $d\omega$ we first note that for $\omega = fdx + gdy + hdz$:

$$\begin{aligned} d\omega &= (f_x dx + f_y dy + f_z dz) \wedge dx + (g_x dx + g_y dy + g_z dz) \wedge dy + (h_x dx + h_y dy + h_z dz) \wedge dz \\ &= f_y dy \wedge dx + f_z dz \wedge dx + g_x dx \wedge dy + g_z dz \wedge dy + h_x dx \wedge dz + h_y dy \wedge dz \\ &= (h_y - g_z) dy \wedge dz - (h_x - f_z) dz \wedge dx + (g_x - f_y) dx \wedge dy \end{aligned}$$

Thus, for our case:

$$d\omega = dz \wedge dx$$

□

Proposition 5.45. (i.) The exterior differentiation $d : \Omega^*(U) \rightarrow \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

(ii.) $d^2 = 0$

(iii.) If $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$

Proposition 5.46. *[Characterization of the exterior derivative]*

The three properties of the previous proposition uniquely characterize exterior differentiation on an open set $U \in \mathbb{R}^n$.

Thus, if a map $D : \Omega^*(U) \rightarrow \Omega^*(U)$ satisfies the above three properties, then $D = d$.
proof here.

5.4 The de Rham Complex and Cohomology

Definition 5.47. [Closed / Exact Form]

A k -form ω on U is *closed* if $d\omega = 0$. It is *exact* if there is a $(k-1)$ -form τ such that $\omega = d\tau$ on U .

Remark 5.48. Since $d(d\tau) = 0$, every exact form is closed.

Definition 5.49. [Cochain Complex]

A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \rightarrow V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex* or a *Cochain complex*.

Definition 5.50. [de Rham complex]

For any open subset U of $\mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the *de Rham complex* of U :

$$0 \rightarrow \Omega^0(U) \rightarrow \Omega^1(U) \rightarrow \Omega^2(U) \rightarrow \dots$$

The closed forms are precisely the elements of the kernel of d , and the exact forms are the elements of the image of d .

Definition 5.51. [de Rham Cohomology - From the Differential Geometry Perspective]

One measure of the failure of a closed k -form to be exact is the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}$$

called the k th de Rham cohomology of U .

5.5 Čech Cohomology

In seeing the construction for the De Rham complex we now look at another example of a cohomology theory, called Čech Cohomology. Čech Cohomology is a completely combinatoric construction, however, some category theory and algebraic geometry definitions may mask the simplicity of the construction. For the purposes of this paper I will outline the construction of the Čech Complex, where one may suspend their belief of the applications to algebraic geometry. Only vague familiarity with functoriality is assumed. Any additional definitions from category theory or algebraic geometry will be introduced as needed.

Motivation

Here X is a Riemann surface, i.e., a complex manifold which has both a differential structure (real compatible charts with diffeomorphic transition functions) and a complex structure (complex compatible charts with holomorphic transition functions).

Definition 5.52 (Presheaf). Let (X, τ) be a topological space. (τ here denotes the topology on X)

A *presheaf*, F , on X is the collection of the following data:

- 1) $\forall U \in \tau$, we assign a set $F(U)$, whose elements are called the *sections* of F over U . *Note that when we say "global sections of F " we are simply referring to the sections of F over the entire space X .*
- 2) For $V, U \in \tau$ such that $V \subset U$, we assign a restriction functor via contravariance, $\text{res}_{V,U} : F(U) \rightarrow F(V)$.

Remark 5.53. The restriction functors above are assumed to satisfy all functoriality axioms.

Definition 5.54 (Sheaves). A *sheaf* is a presheaf which satisfies two additional properties:

- 1) Given $U \in \tau$, $\{U_i\}_{i \in I}$ an open cover of U , and $s, t \in F(U)$, then if $\forall i \in I$, $s|_{U_i} = t|_{U_i}$, it must follow that $s = t$.
- 2) Given a family of sections $\{s_i\}_{i \in I}$ where $s_i \in F(U_i)$, then if all pairs of sections agree on the overlap of domain, i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, $\forall i, j \in I$, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Remark 5.55. The first of the above additional properties of sheaves allows for a well-defined local approximation of X and the second additional property allows us to glue together sections of an open cover uniquely.

Remark 5.56. In practice we usually run into very beautiful short exact sequences of Sheaves:

$$O \longrightarrow K \hookrightarrow F \longrightarrow G \longrightarrow O$$

However, this doesn't lead to a short exact sequence of global sections

$$O \longrightarrow K(X) \hookrightarrow F(X) \longrightarrow G(X) \not\rightarrow O$$

Thus, the global section functor is NOT right exact. However, not all is lost. We want to construct our cohomology theory in such a way that although we do not have a short exact sequence of sheaves, we can extend to a long exact sequence of cohomology:

$$O \longrightarrow K(X) \hookrightarrow F(X) \longrightarrow G(X) \longrightarrow H^1(X, K)$$

Remark 5.57. *Why do we have this failure? Why is the global section functor not right exact?*

For map on sheaves, if it is injective, then it is injective on sections:

$$F \hookrightarrow G \implies F(U) \hookrightarrow G(U)$$

$$\text{where, } \begin{array}{ccc} i: F(U) & \longrightarrow & G(U) \\ \wr & & \wr \\ f & \longmapsto & 0 \end{array}$$

However, if it is surjective, then it doesn't necessarily mean it will be surjective on sections:

$$F \twoheadrightarrow G \not\Rightarrow F(U) \twoheadrightarrow G(U)$$

Why? Injectivity and surjectivity on sheaves are equivalent to injectivity and surjectivity on stalks.

Let's assume injectivity on stalks: $F_p \hookrightarrow G_p$. Thus, we say that $F(U) \hookrightarrow G(U)$. let f be a section over U of F such that $f \mapsto 0$. Because the map $F_p \hookrightarrow G_p$ is injective on stalks then there must be a cover of U , call it $\{u_i\}_{i \in I}$, such that locally $f|_{u_i} = 0$. But by sheaf axioms each restriction glues to f and we have $f = 0$. Good! This tells us that injectivity on stalks imply injectivity on sections.

Now assume surjectivity on stalks: $F_p \twoheadrightarrow G_p$. Let's say we have a section g on U of G . Therefore, we can find a cover $\{u_i\}_{i \in I}$ of U such that locally g has a preimage in f . Thus, $f_i \mapsto g|_{u_i}$.

However, unlike in the injective case, we cannot say that the set $f_{i \in I}$ of local preimages can glue together because they may not be compatible, i.e., they may not agree on intersection:

$$f_i|_{u_i \cap u_j} \neq f_j|_{u_i \cap u_j} \implies \text{cannot glue } f_i \text{ together to a global preimage!}$$

□

Remark 5.58. *Thus, we want to define $H^1(X, K)$ in such a way which measures the extent to which local pre-images fail to glue to a global pre-image. i.e. how badly does the surjectivity on stalk not imply surjectivity on sections?*

Recall our long exact sequence of cohomology:

$$O \longrightarrow K(X) \hookrightarrow F(X) \xrightarrow{\delta_1} G(X) \xrightarrow{\delta_2} H^1(X, K)$$

We want to define some equivalence relation $\sim$ such that:

$$\begin{array}{ccc} \delta_2: G(x) & \longrightarrow & H^1(X, K) \\ \downarrow & & \downarrow \\ g & \longmapsto & (f_i - f_j)_{u_{ij}} \end{array} / \sim$$

The intuition for this is because we want the image of this map π to capture compatibility of local preimages, thus we can think of elements of $H^1(X, K)$ as tuples where in the intersection $u_{ij} = u_i \cap u_j$ we take the difference $f_i - f_j$

We want an equivalence relation, $\sim$, such that g is in the kernel of δ_2 only if it is in the image of δ_1 . Since if g is in the image of δ_1 it has a global preimage!

So what should $\sim$ be? What should we be quotienting by? Well, naturally one may say that we should just require that the preimages agree on intersection. Thus,

$$f_i|_{u_{ij}} = f_j|_{u_{ij}}$$

This is sufficient to conclude the existence of global preimages, however, this isn't really necessary. The preimages f_i, f_j are not unique. On each u_i if f_i is a preimage, then if we add another element in the kernel of the map restricted to u_i , we get another preimage on u_i . So just because the preimages we have chosen for the cover $u_{i \in I}$ don't glue g could still have a global preimage if we pick a different cover!

So the condition we will use to guarantee that g has a global pre-image would be if there exists an element $k_i \in K(u_i)$ such that if we modify the local preimages of u_i by k_i , we get a compatible set of preimages which will glue together to form a global preimage:

$$f_i + k_i|_{u_{ij}} = f_j + k_j|_{u_{ij}}$$

Thus, we can write:

$$f_i - f_j|_{u_{ij}} = k_j - k_i|_{u_{ij}}$$

and define $h_{ij} = f_i - f_j$ to give us the elements in $H^1(X, K)$:

$$H^1(X, K) = \{(h_{ij})_{u_{ij}}\} / \{(k_i - k_j)_{u_{ij}}\}$$

where $k_i \in K(u_i)$.

However, there are some things here to be worried about. This construction seems to depend on the choice of cover $\{u_i\}$

Definition

Recall: Our temporary definition of $H^1(X, K)$:

$$O \longrightarrow K(X) \hookrightarrow F(X) \xrightarrow{\delta_1} G(X) \xrightarrow{\delta_2} H^1(X, K)$$

For the above sequence to be exact we need $\text{Ker}(\delta_2) = \text{Im}(\delta_1)$, i.e., for $g \in G(X)$, if $g \mapsto 0 \iff g \in \text{Im}(\delta_1)$, thus, g has a global preimage.

Thus, we take the temporary definition that $H^1(X, K)$ consists of elements h_{ij} of the kernel sheaf $K(u_{ij})$, which we pick for each u_{ij} , where we identify elements h_{ij} , which come from a difference of sections $k_i - k_j$, where k_i, k_j are also elements of the kernel sheaf over u_i or u_j respectively.

This ensures that $g \mapsto 0 \iff f_i - f_j|_{u_{ij}} = 0$ and we have the compatibility condition sufficient to conclude that g has a global preimage.

Remark 5.59. Note that cohomology ALWAYS comes from a cochain complex. Thus, to define a cohomology we simply need to give the cochain complex associated with it.

Let $U = \{u_i\}$ and $X = \bigcup_i u_i$. A cochain complex is a chain of groups and maps such that $\delta_{n+1} \circ \delta_n = 0$:

$$C^0(U, K) \xrightarrow{\delta_0} C^1(U, K) \xrightarrow{\delta_1} C^2(U, K) \xrightarrow{\delta_2} \dots$$

Notice that $\delta_{n+1} \circ \delta_n = 0$ tells us that $\text{Ker}(\delta_{n+1}) \subseteq \text{Im}(\delta_n)$.

Definition 5.60. The n th order Čech cohomology of a sheaf K with respect to a covering U is given by:

$$H^n(U, K) = \text{Ker}(\delta_n) / \text{Im}(\delta_{n-1})$$

Where we define the elements in the n -cochain as:

For the zero cochain C^0 we have tuples where we pick a section of the kernel sheaf $k_i \in K(u_i)$ for each u_i , denoted $(k_i)_{u_i}$. For the one cochain we have tuples where we pick a section of the kernel sheaf $k_{ij} \in K(u_{ij})$ for the intersection of u_{ij} , denoted $(k_{ij})_{u_{ij}}$. And similarly for the two cochain we have tuples where we pick a section of the kernel sheaf $k_{ijk} \in K(u_{ijk})$ for the intersection of u_{ijk} , denoted $(k_{ijk})_{u_{ijk}}$.

We define the maps δ_n between cochains as follows

$$\begin{array}{ccc} \delta_0: C^0(U, K) & \longrightarrow & C^1(U, K) \\ \downarrow \Psi & & \downarrow \Psi \\ (k_i)_{u_i} & \longmapsto & (k_i - k_j)_{u_{ij}} \end{array}$$

$$\begin{array}{ccc} \delta_1: C^1(U, K) & \longrightarrow & C^2(U, K) \\ \downarrow \wr & & \downarrow \wr \\ (k_{ij})_{u_{ij}} & \longmapsto & (k_{ij} - k_{jk} + k_{ki})_{u_{ijk}} \end{array}$$

Sanity Check! Compute $H^0(U, K)$.

$$H^0(U, K) = \text{Ker}(\delta_0)$$

The kernel of the map δ_0 consists of tuples $(k_i)_{u_{ij}}$ such that $\delta_0((k_i)_{u_{ij}}) = k_i - k_j = 0$. In other words they are tuples which are compatible on u_{ij} (i.e. $k_i = k_j$ on u_{ij}), thus, they must glue together to make a global section! Furthermore:

$$H^0(U, K) = \text{Ker}(\delta_0) = K(X)$$

Definition 5.61. We define a Cech cohomology which is **INDEPENDENT** of covering as the following dirac limit:

$$\check{H}^n(X, K) = \varinjlim_U \check{H}^n(U, K)$$

Rigorous definition of Dirac Limit: What is the ordering on U ?

First note that U is a refinement of V , denoted $U < V$ if for all $U_i \in U$, there is some $V_{r(j)} \in V$ such that $U_i \subseteq V_{r(j)}$. Where $r : \text{index of } U \rightarrow \text{index of } V$ is a refinement map which sends the index of a set $U_i \in U$ to the set $V_{r(j)} \in V$ which contains it (in this case $i \mapsto r(i)$). In the example below r would be a map such that $0, 1 \mapsto 0$ and $2 \mapsto 1$.

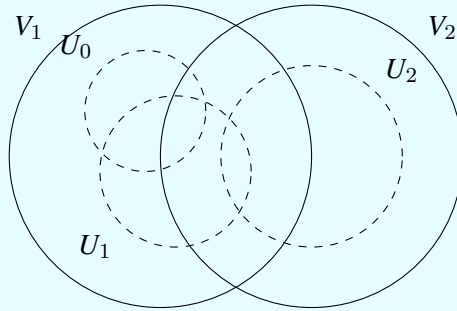


Figure 13

Once we have this refining map r we can induce a map on the cohomology:

$$\begin{array}{ccc} \pi^n : \check{H}^n(U, K) & \longrightarrow & \check{H}^n(V, K) \\ \downarrow \Psi & & \downarrow \Psi \\ (k_{i_0 \dots i_n}) & \longmapsto & (k_{r(i_0) \dots r(i_n)}) \end{array}$$

For the example coverings in figure 1 above the π^0 map would give us:

$$\begin{array}{ccc} \pi^0 : \check{H}^0(V, K) & \longrightarrow & \check{H}^0(U, K) \\ \downarrow \Psi & & \downarrow \Psi \\ (f_1, f_2) & \longmapsto & (f_1|_{U_1}, f_1|_{U_2}, f_2|_{U_3}) \end{array}$$

This map in general is well-defined and independent of the choice of the refining map. Thus, this will form a dirac system where the ordering of the coverings is by refinement.

Note that this notion of refinement is not the usual sense that we use for refining a partition for an Riemann integration. Instead it is like "refining" the sizes of the sets smaller and smaller rather than the total number of sets. In the picture below, $U < U'$ (insert pic from part 4)

Loose Definition of Dirac Limit: You can think of the dirac limit roughly as the "union" of the cohomologies $\check{H}^n(U, K)$ over coverings U .

How do we define "union"? Let's look at the $n = 0$ case: $\check{H}^0(U, K) \cup \check{H}^0(V, K)$, where the covering U consists of two open sets U_1 and U_2 and the covering V consists of just the point $\{x\}$.

Thus, elements in $\check{H}^0(U, K)$ are equivalence classes of tuples $[(s_1, s_2)]$ where $s_i \in K(U_i)$. Where as in $\check{H}^0(V, K)$ the elements are the equivalence class of a global section of K , i.e. $[(f)]$, where $f \in K(x)$. A natural choice of "union" is the disjoint union, where we are preserving the information from the cohomology associated with each covering. But this is redundant since for each global section $f \in K$ we have a pair of local sections restricted to U_1 and U_2 , denoted $[(f|_{U_1}, f|_{U_2})]$, and the disjoint union would take $[(f)]$ and $[(f|_{U_1}, f|_{U_2})]$ as different elements!

Another logical option to consider is to take the "union" to be the disjoint union quotiented by a relation $\sim$, where we identify $(s_1, s_2) \sim (f)$ for any tuple $(s_1, s_2) \in C^0(U, K) \times C^0(U, K)$ which is compatible with $(f|_{U_1}, f|_{U_2})$.

Note that if V is a covering consisting of two open sets, V_1, V_2 , and U is a covering consisting of two open sets, U_1, U_2, U_3 such that $U_1, U_2 \subseteq V_0$ and $U_3 \subseteq V_1$ i.e. **figure 1 above**. Then every element in $\check{H}^0(V, K)$ is some pair (f_1, f_2) which is identified to a triple $(f_0|_{U_0}, f_0|_{U_1}, f_1|_{U_2}) = (s_0, s_1, s_2)$.

Also notice that in the case of $n = 1$ this notion of "union" is no longer an analogy and we get an actual union:

$$\check{H}^1(X, K) = \bigcup \check{H}^1(U, K) = \varinjlim_U \check{H}^1(U, K)$$

H^0 and H^1

Why Care? - Zeroth Order Cohomology: H^0

Claim: No matter what covering U of X we choose, $H^0(U, K)$ is isomorphic to the group of global sections, $K(X)$.

proof. First recall what the elements of $H^0(U, K)$. If $U = \{u_i\}_i$, then the elements of $H^0(U, K)$ generated by the 0 cochain are of the form of tuples $[(f_i)_{u_i}]$ where f_i is a section over U_i . Since this is a 0 cocycle, $[(f_i)_{u_i}]$ must lie in the kernel of the coboundary map δ_0 , i.e.:

$$\delta_0([(f_i)_{u_i}]) = 0$$

This is then just saying that each f_i is compatible by definition of δ_0 ! Since the set $\{f_i\}$ are compatible, by sheaf axioms they can glue together to make a unique global section and we have thus the isomorphism:

$$\check{H}^0(U, K) \cong K(X)$$

Thus, since this is simply a restriction on the dirac limit, we have the isomorphism:

$$\varinjlim_U \check{H}^0(U, K) = \check{H}^0(X, K) \cong K(X)$$

□

Why Care? - The First Order Cohomology: H^1

Claim: The first order Čech cohomology $\check{H}^1(X, K)$ is the union over covers U of X of first order Čech cohomologies $\check{H}^1(U, K)$, i.e.

$$\check{H}^1(X, K) = \varinjlim_U \check{H}^1(U, K) = \bigcup_U \check{H}^1(U, K)$$

Recall: The cohomology groups are the disjoint union of the cohomology groups with respect to a covering quotiented by refinements (refer to the loose definition of Dirac Limit section). But to do this we need to identify these refinement maps between coverings.

However, in the case of degree 1, the refinement maps turn out to be injective!

$$\begin{array}{ccc} r: \check{H}^1(U, K) & \longrightarrow & \check{H}^1(V, K) \\ \downarrow \Psi & & \downarrow \Psi \\ (k_{ab})_{u_{ab}} & \longmapsto & 0 \end{array}$$

Thus, we are only identifying cocycles that are already equal! So the dirac limit is indeed just a normal union of all degree one cohomologies with a specific covering. The injections allow you to view, $\check{H}^1(U, K)$ as a subgroup of $\check{H}^1(V, K)$ so you are essentially identifying $\check{H}^1(U, K)$ with itself.

proof. We want to show that the maps induced by refinements are actually injective. Thus, it suffices to show $H^1(U, K) \hookrightarrow H^1(V, K)$ so long as $V < U$. Since $V < U$, we have some refining map r associated with this refinement. In the picture below we see $r(i) = \alpha$ and $r(j) = \beta$.

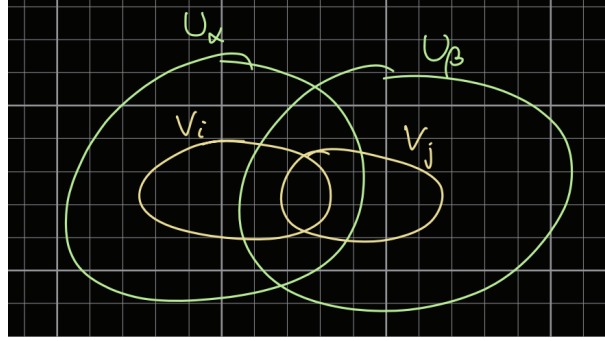


Figure 14: Example Refinement

The refinement map then induces a map between respective cohomologies as follows:

$$\begin{array}{ccc} \pi: \check{H}^1(U, K) & \longrightarrow & \check{H}^1(V, K) \\ \downarrow \psi & & \downarrow \psi \\ (k_{\alpha\beta})_{U_{\alpha\beta}} & \longmapsto & 0 \end{array}$$

Where since we are dealing with one cochains: $k_{\alpha\beta}$ is a section over the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and the map π simply takes $(k_{\alpha\beta})$ and sends it to $(k_{\alpha\beta}|_{V_{ij}})$ i.e. $(k_{r(i)} \ r(j))$ as shown in the picture below:

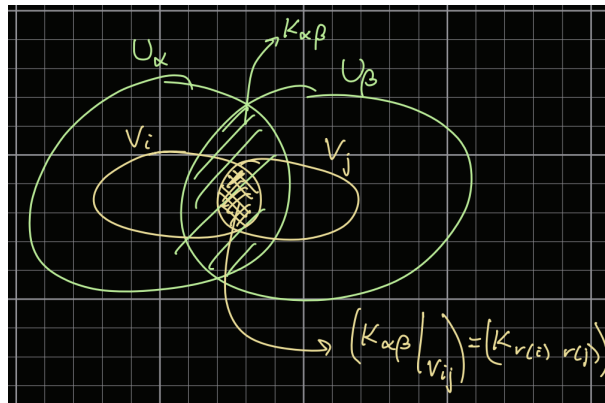


Figure 15: Visualizations of the Cochain calculation

However, note that the element $k_{\alpha\beta}$ is not only a one cochain, but a one cocycle. Thus, the element $k_{\alpha\beta}$ should lie in the kernel of δ_1 , thus:

$$\begin{array}{ccc} \delta_1: U_{\alpha\beta} \subset C^1(U, K) & \longrightarrow & U_{\alpha\beta\gamma} \subset C^2(U, K) \\ \downarrow \Psi & & \downarrow \Psi \\ (k_{\alpha\beta}) & \longmapsto & (k_{\beta\gamma} - k_{\alpha\gamma} + k_{\alpha\beta}) \end{array}$$

where since $k_{\alpha\beta}$ is in the kernel of δ_1 ,

$$\delta_1(k_{\alpha\beta}) = (k_{\beta\gamma} - k_{\alpha\gamma} + k_{\alpha\beta}) = 0$$

Thus, $\forall \alpha, \beta, \gamma$ such that $U_{\alpha\beta\gamma} \neq \emptyset$, $k_{\alpha\beta} = k_{\alpha\gamma} - k_{\beta\gamma}$.

To prove injectivity suppose we have a one cocycle, $k_{\alpha\beta}$, which maps to 0 by δ_1 . It suffices to show that $k_{\alpha\beta}$ is of the form $(h_\beta - h_\alpha)_{U_{\alpha\beta}}$ for sections h_i of $K(U_\alpha)$.

Since δ_1 maps the cocycle $k_{\alpha\beta}$ to 0 it maps cocycles to coboundaries! Thus, there must exist sections f_i on V_i and f_j on V_j such that:

$$k_{r(i) \ r(j)} = f_j - f_i \text{ on } V_{ij}$$

From the cocycle condition that $k_{\alpha\beta} = k_{\alpha\gamma} - k_{\beta\gamma}$, we see the above equation tells us:

$$k_{r(i) \ r(j)} = k_{\alpha r(j)} - k_{\alpha r(i)} \text{ on } V_{ij} \cap U_\alpha$$

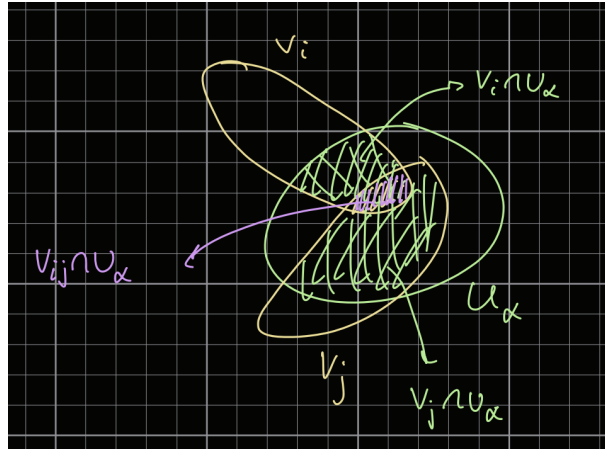


Figure 16: Keeping Track of Intersections

This then implies that:

$$k_{r(i)} r(j) = f_j - f_i \text{ on } V_{ij} \implies f_j - k_{\alpha r(j)} = f_i - k_{\alpha r(i)} \text{ on } V_{ij} \cap U_\alpha$$

Now define $h_{\alpha j} = f_j - k_{\alpha r(j)}$ on $V_j \cap U_\alpha$ and $h_{\alpha i} = f_i - k_{\alpha r(i)}$ on $V_i \cap U_\alpha$! Since all these maps are compatible they glue together to a section h_α !

Thus, to show $k_{\alpha\beta} = h_\beta - h_\alpha$ (and therefore $k_{\alpha\beta}$ is a one coboundary), it suffices to check that h_α and h_β agree locally on $V_i \cap U_{\alpha\beta}$:

$$h_\beta = f_i - k_{\beta r(i)}$$

$$h_\alpha = f_i - k_{\alpha r(i)}$$

$$\begin{aligned} \implies k_{\alpha\beta} &= h_\beta - h_\alpha \\ &= (f_i - k_{\beta r(i)}) - (f_i - k_{\alpha r(i)}) \\ &= k_{\alpha r(i)} - k_{\beta r(i)} \end{aligned}$$

which is just the cocycle condition! Thus, the map π must be injective and $H^1(X, K)$ is a union of all one cohomologies over the coverings U of X . $\square$

References

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